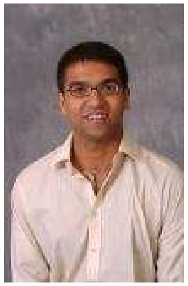
The Disappearing Second Derivative of Quadratics: Perceptual, Mathematical, and Statistical Properties of Judging Dependence on Visual Displays

Saptarshi Guha



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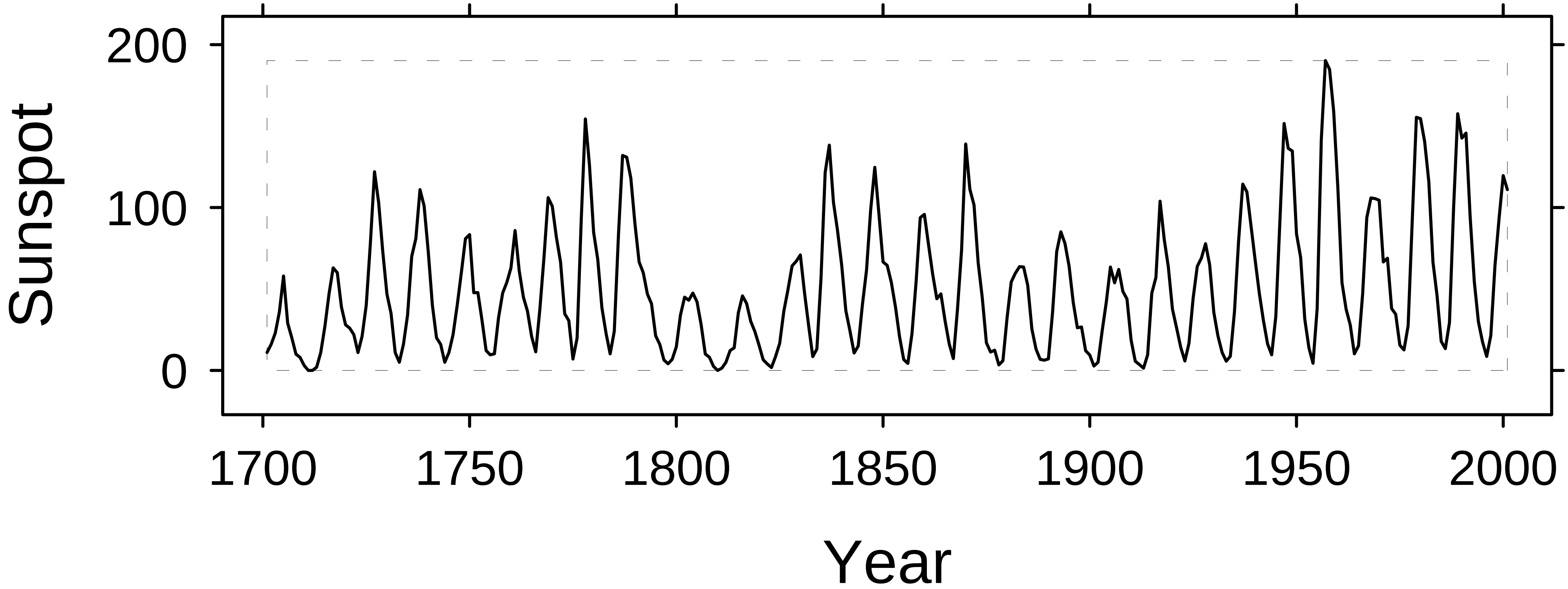
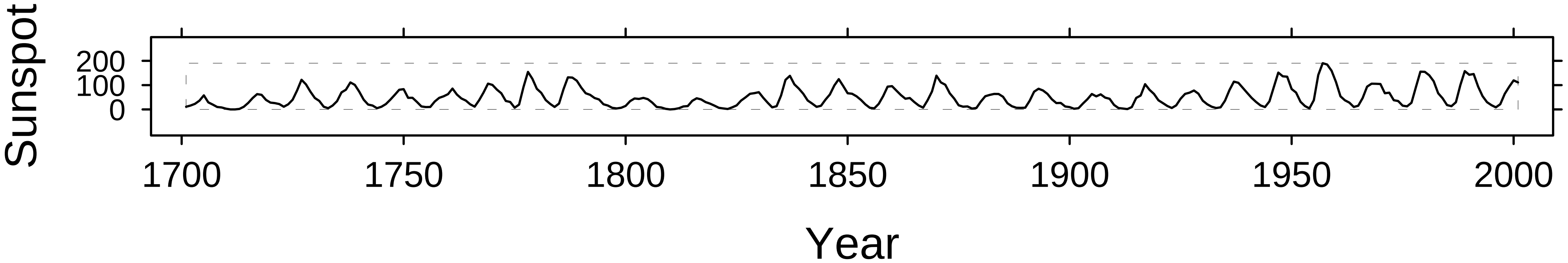
Yearly Sunspot Numbers



11-year cycle is evident

An important phenomenon cannot be readily seen

Yearly Sunspot Numbers



An asymmetry now clearly revealed

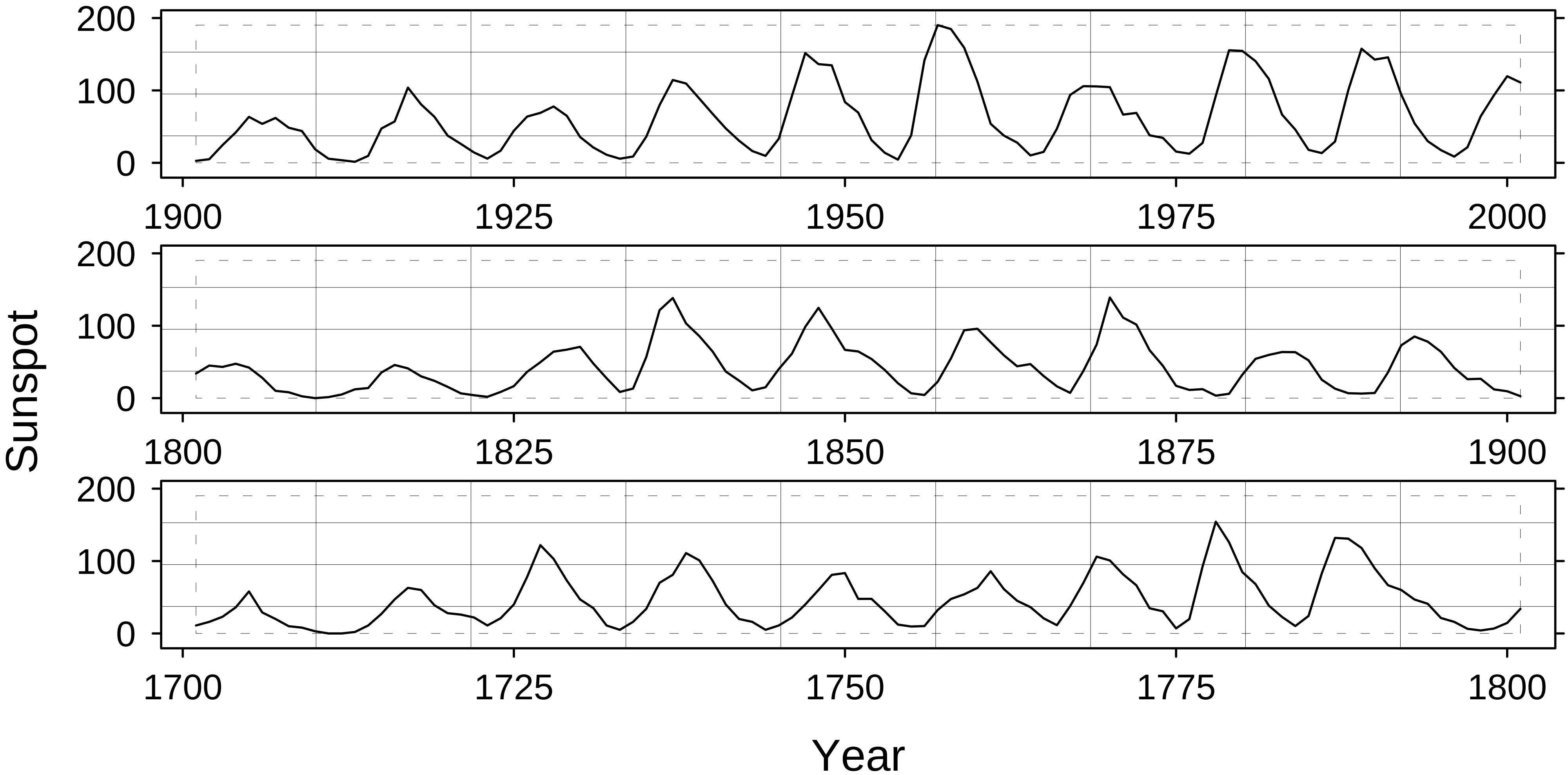
Sunspots tend to rise more rapidly than they fall

Disparity in the absolute slopes for a cycle

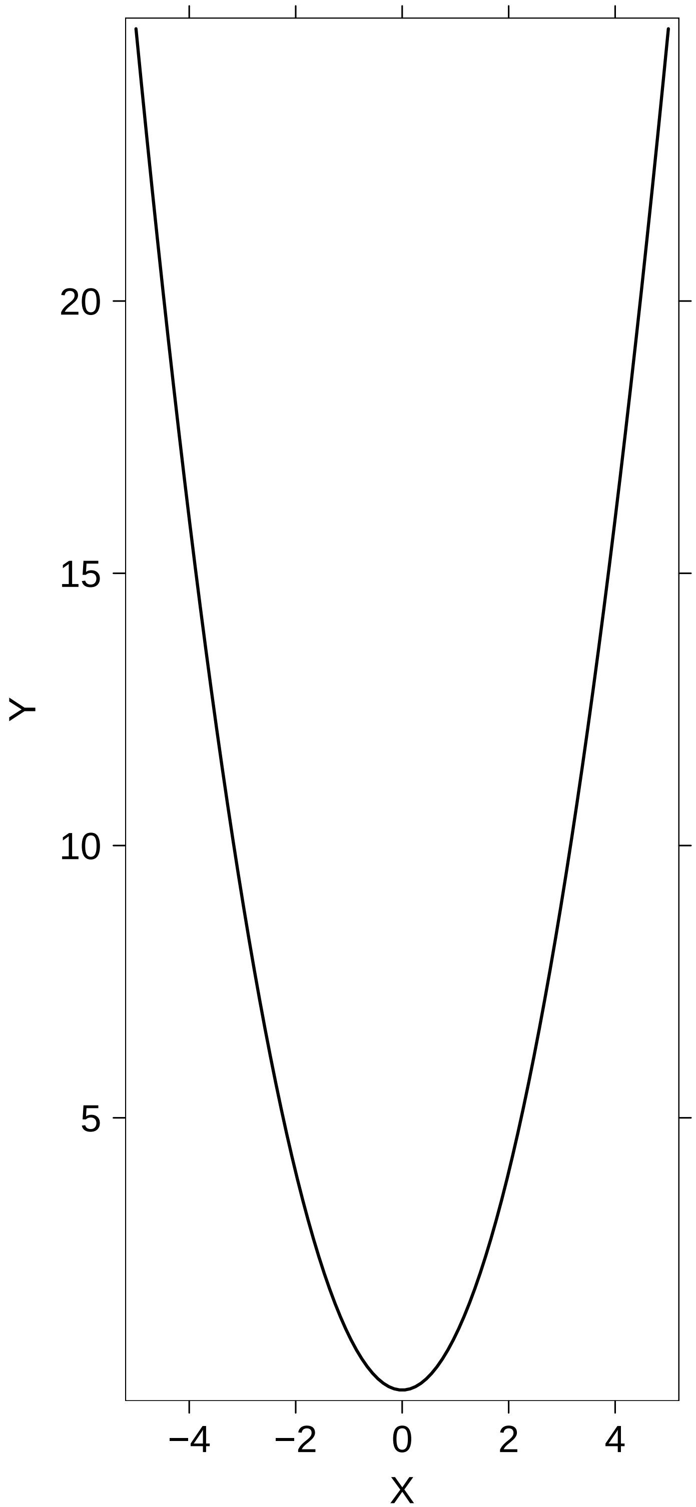
- greatest for the those with the greatest peaks
- lessens as the peak values decrease
- negligible for the lowest peaks

Vertical resolution of display at its limit

Yearly Sunspot Numbers



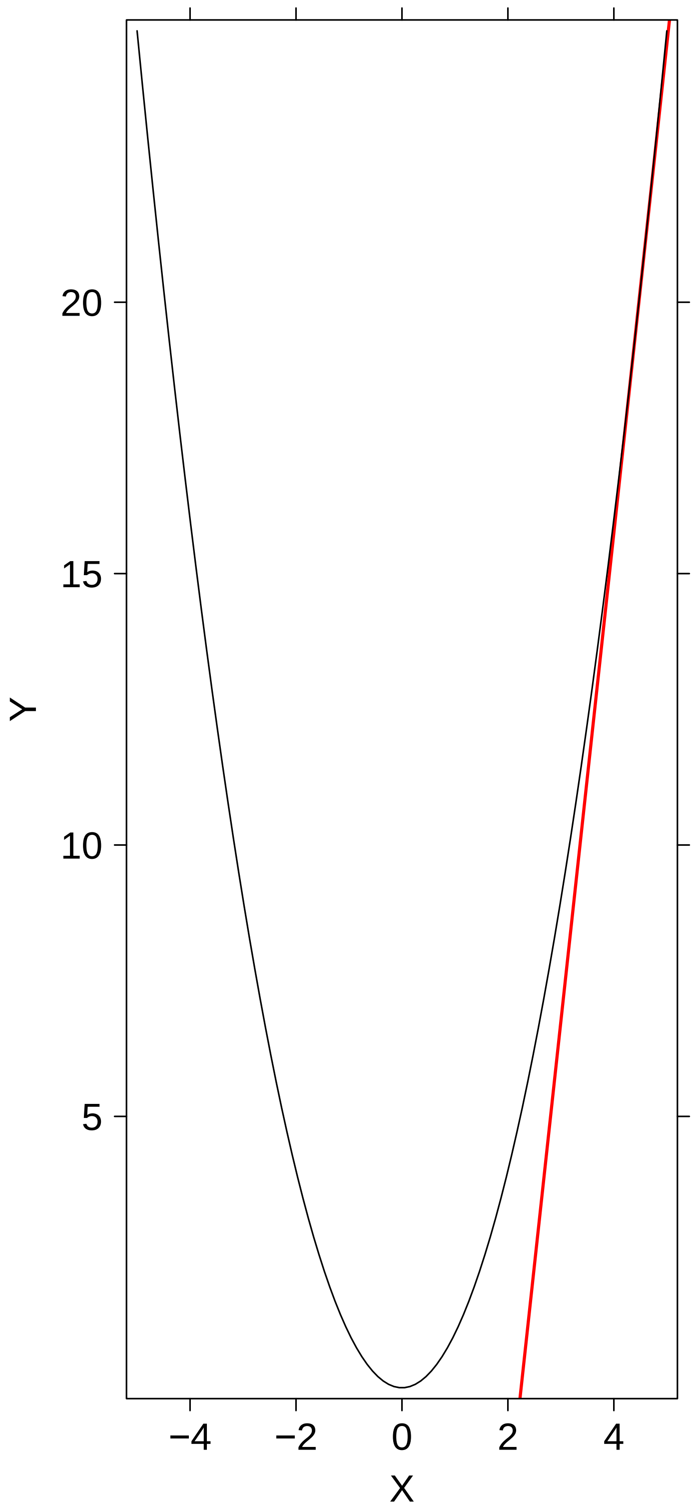
$Y = X^2$



Displayed with vertical-scale units per cm same as horizontal

There is a problem

$Y = X^2$

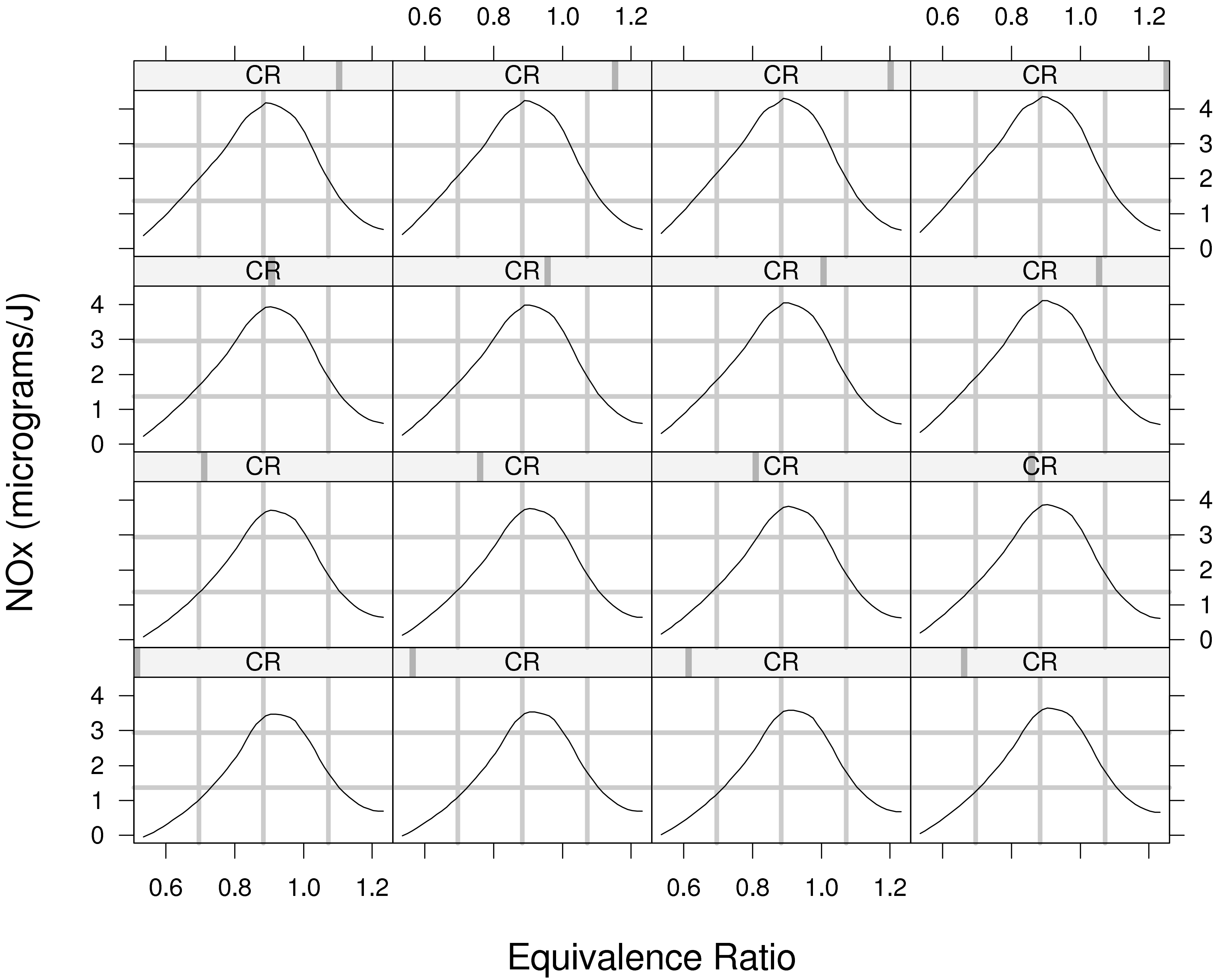


The tangent at $x = 4.5$

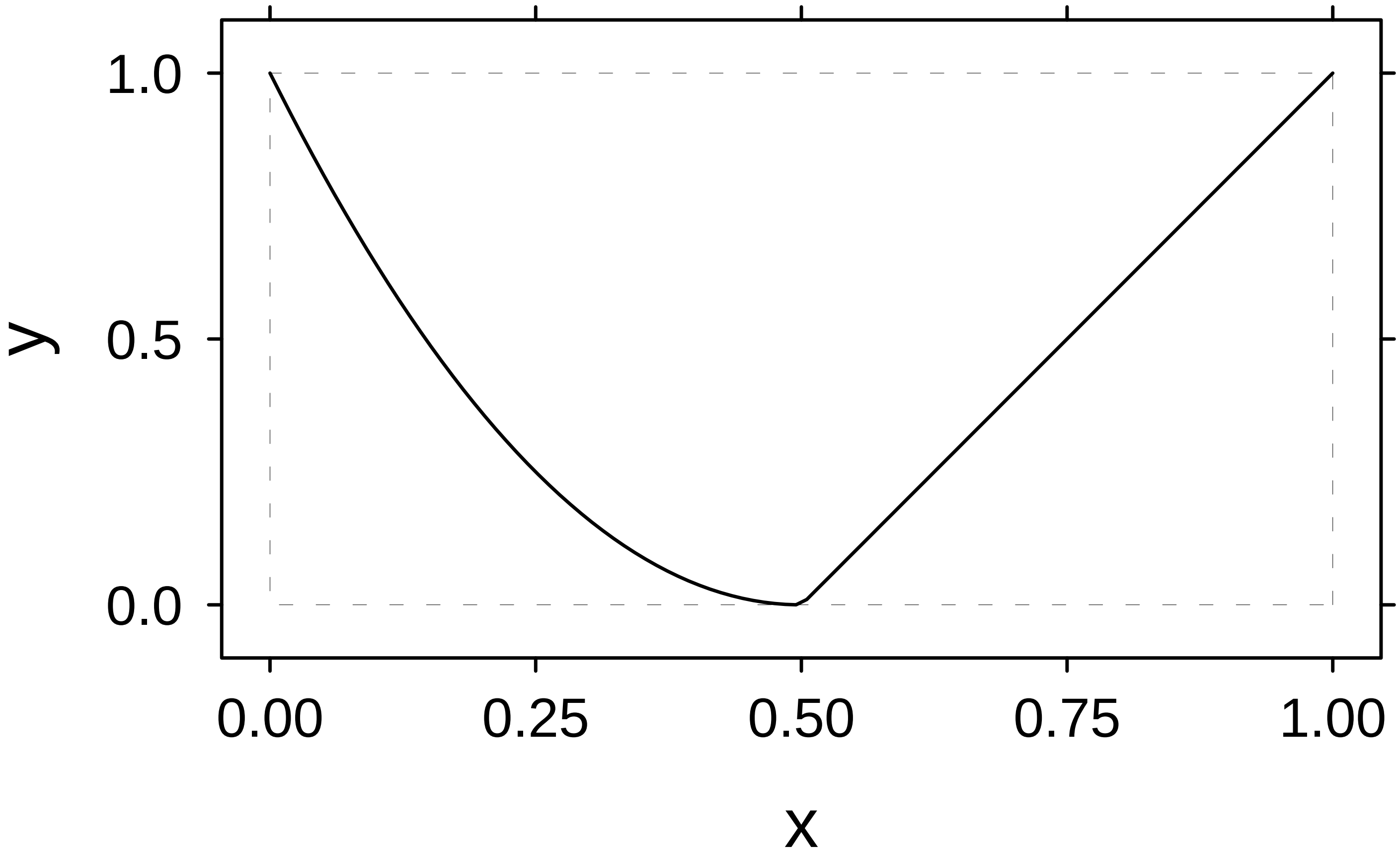
The second derivative is apparent near the origin

It disappears as $|x|$ gets large

Everything Applies to Multiple Curves on Same or Different Panels



Data Rectangle and Segment Aspect Ratio



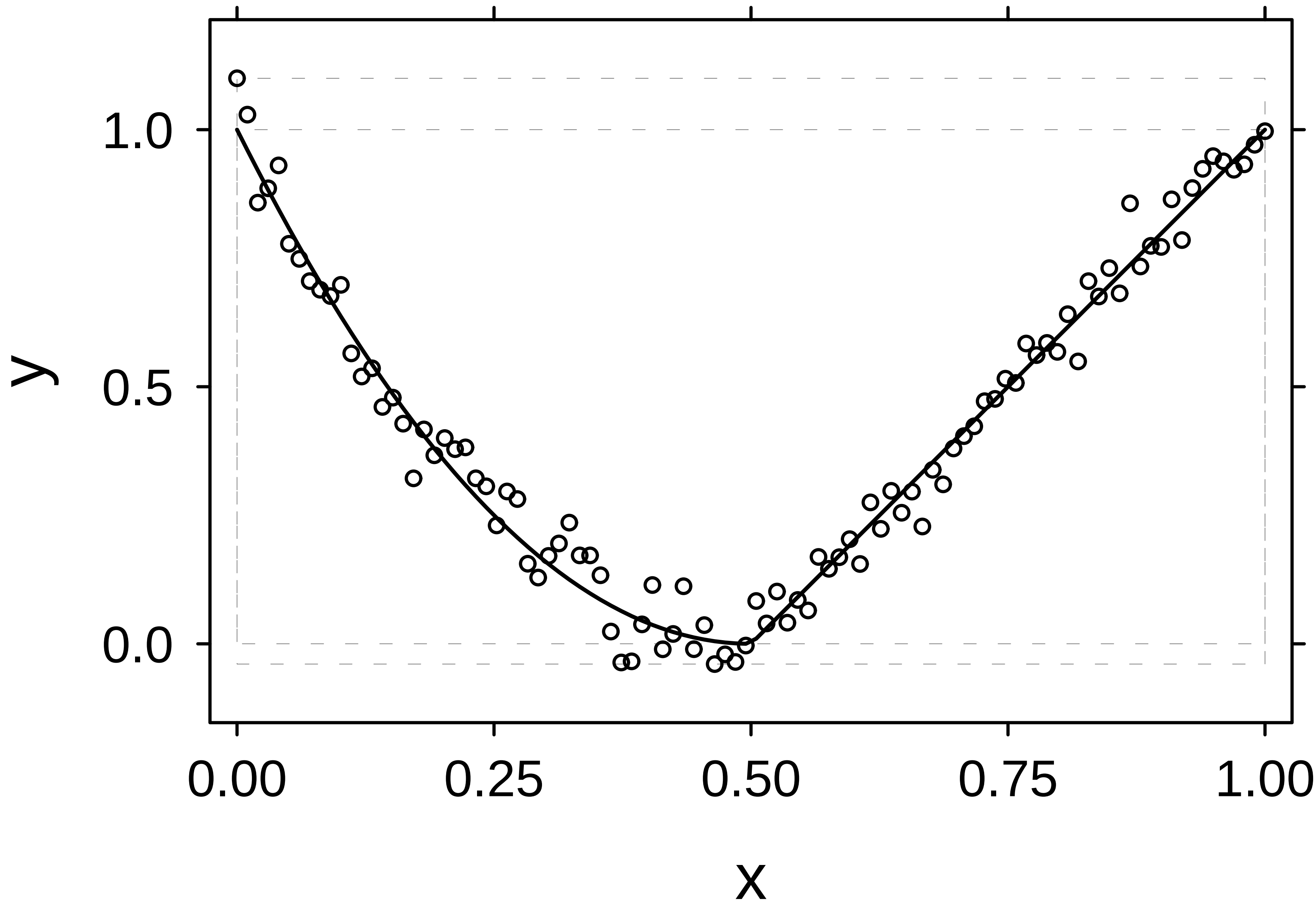
Curve rendered by n line segments

The segment data rectangle just encloses the curve

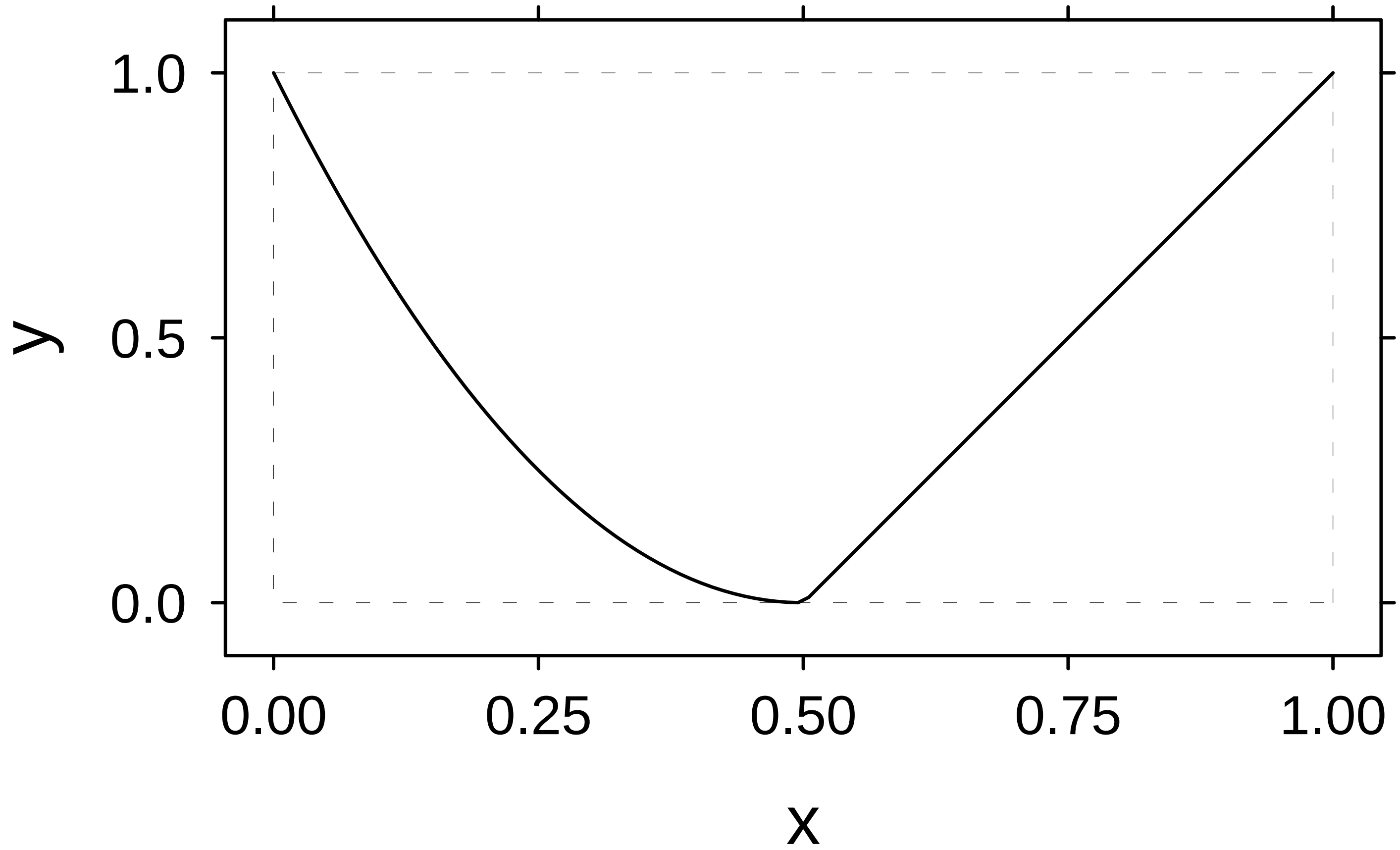
- v = vertical side length in cm
- h = horizontal side length in cm

a = v/h = segment aspect ratio, equal to 1/2 above

Data Other Than Segments



Fundamental Perceptual Task: Visually Decoding Rate of Change

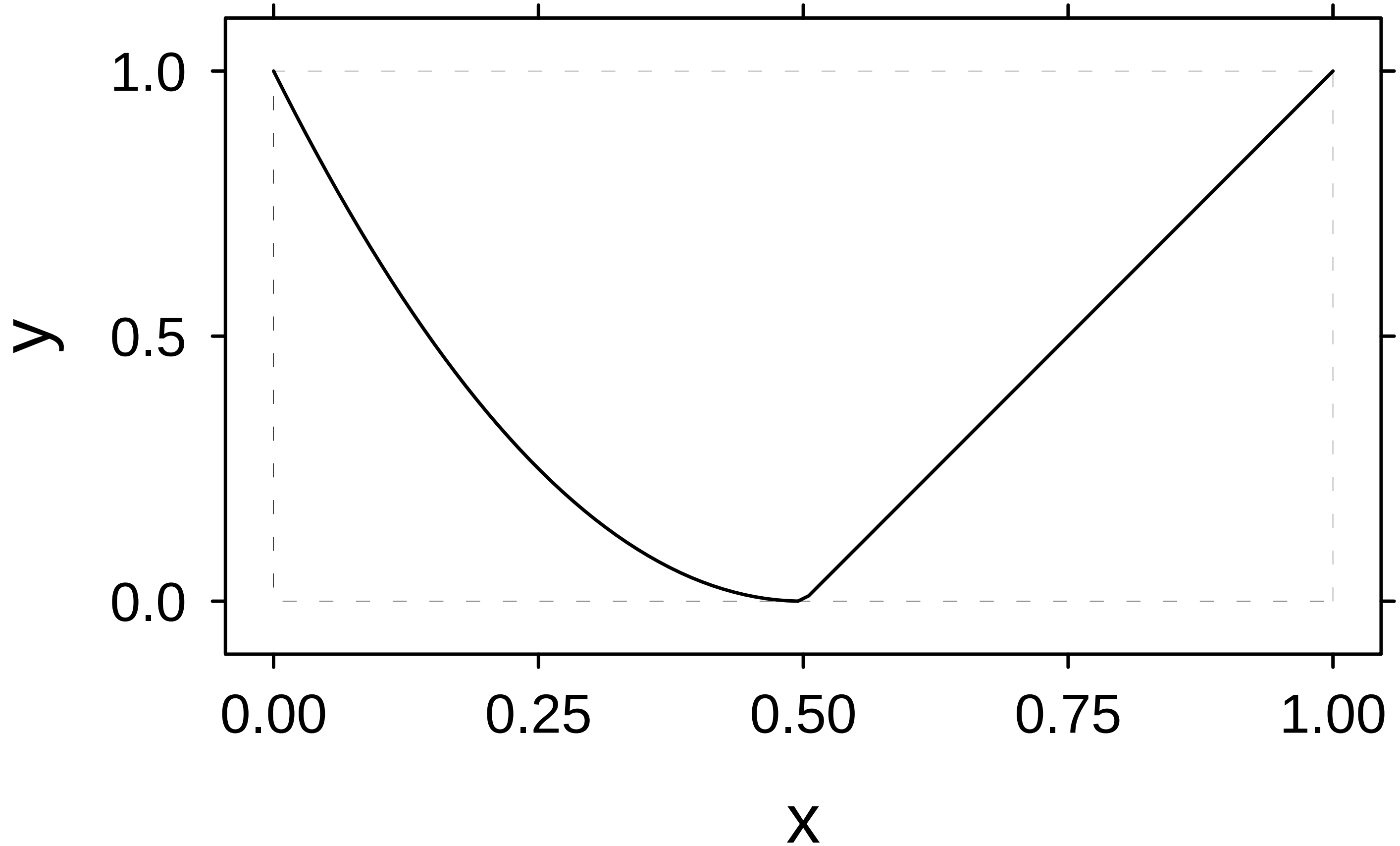


Let s_k for $k = 1$ to n be physical slopes when $a = 1$

Slopes are as_k when aspect ratio is a

Judge slopes of segments, as_k to visually decode rate of change of y with x

a and θ_k



Let $\theta_k = \arctan(as_k)$ be physical orientations ($-90°$ to $90°$)

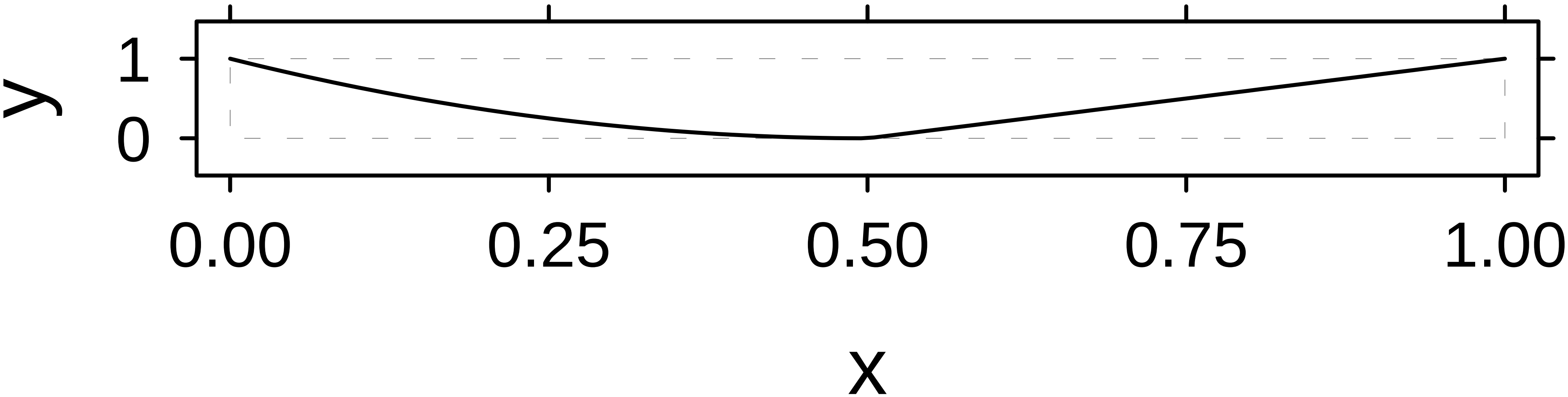
We perceive θ_k, not s_k

a controls orientations

a and θ_k

$\theta_k \longrightarrow 0$ as $a \longrightarrow 0$

a = 1/16

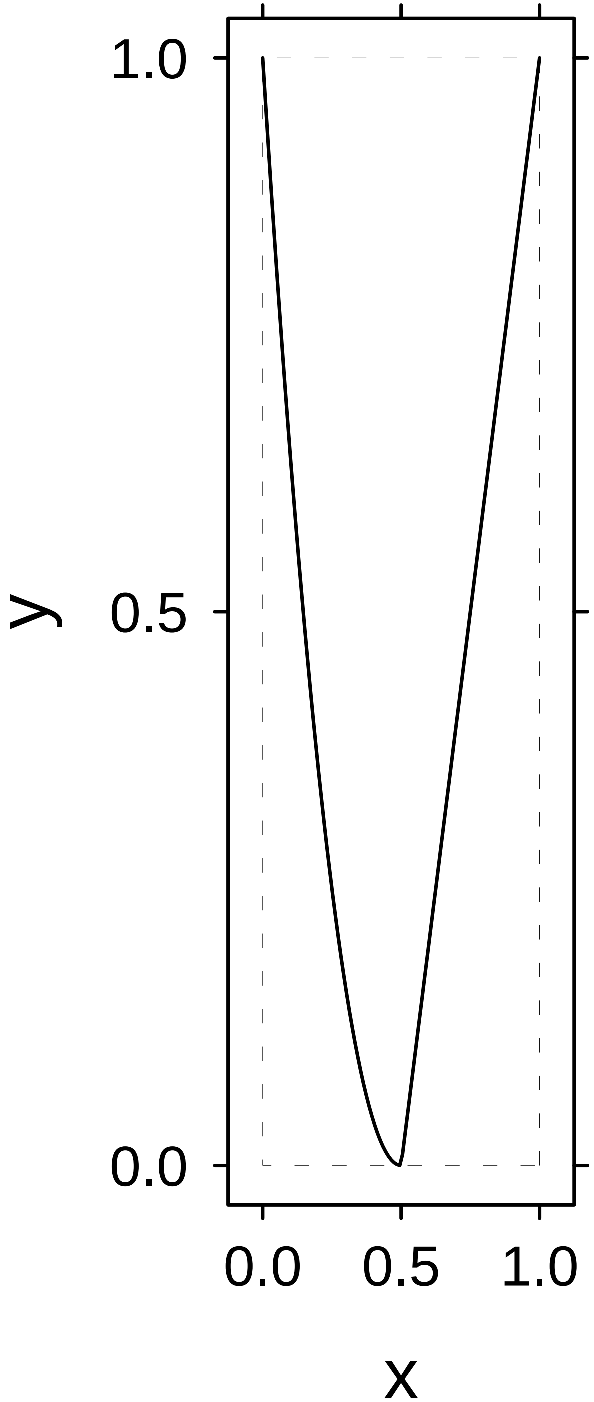


a and θ_k

For $\theta_k > 0, \theta_k \rightarrow 90^\circ$ as $a \rightarrow \infty$

For $\theta_k < 0, \theta_k \rightarrow -90^\circ$ as $a \rightarrow \infty$

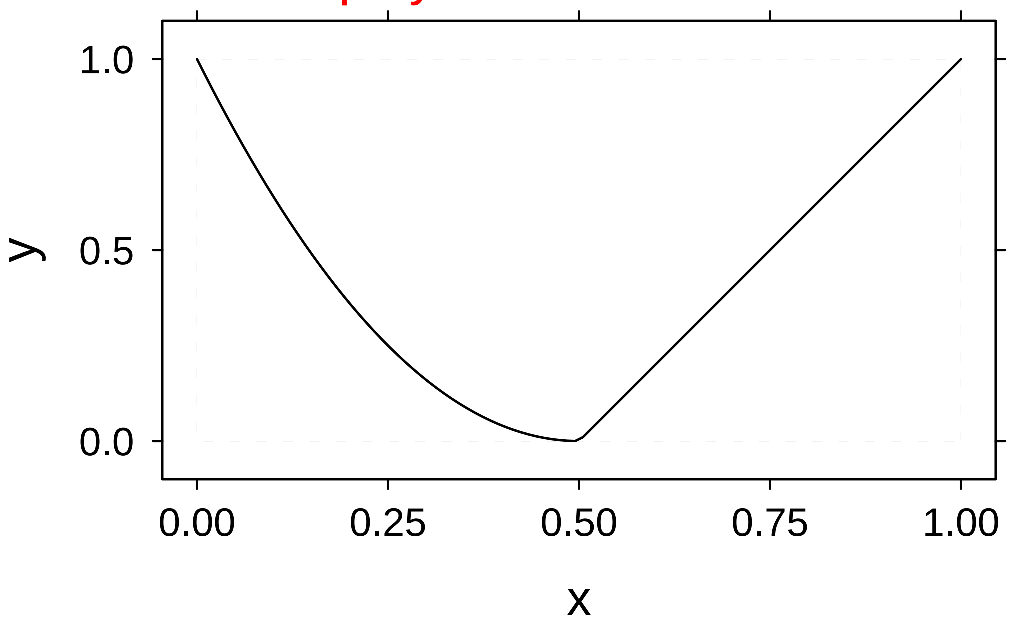
a = 4

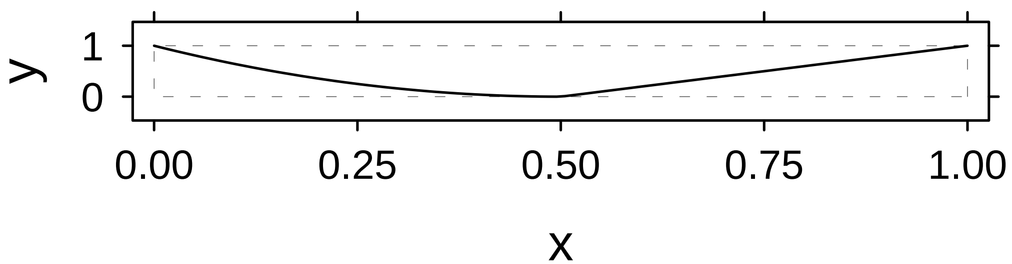


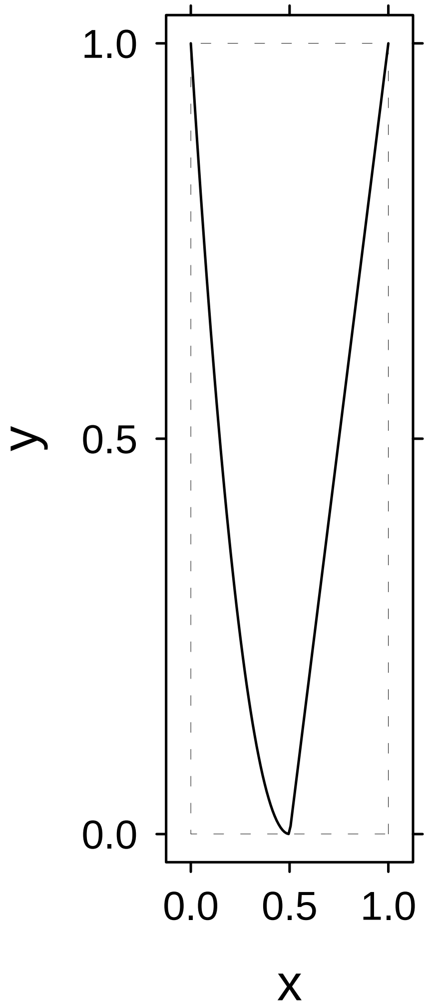
Banking to 45°

Visual decoding of rate of change by judging θ_k optimized when $|\theta_k|$ are centered on 45°

This display is banked to 45°





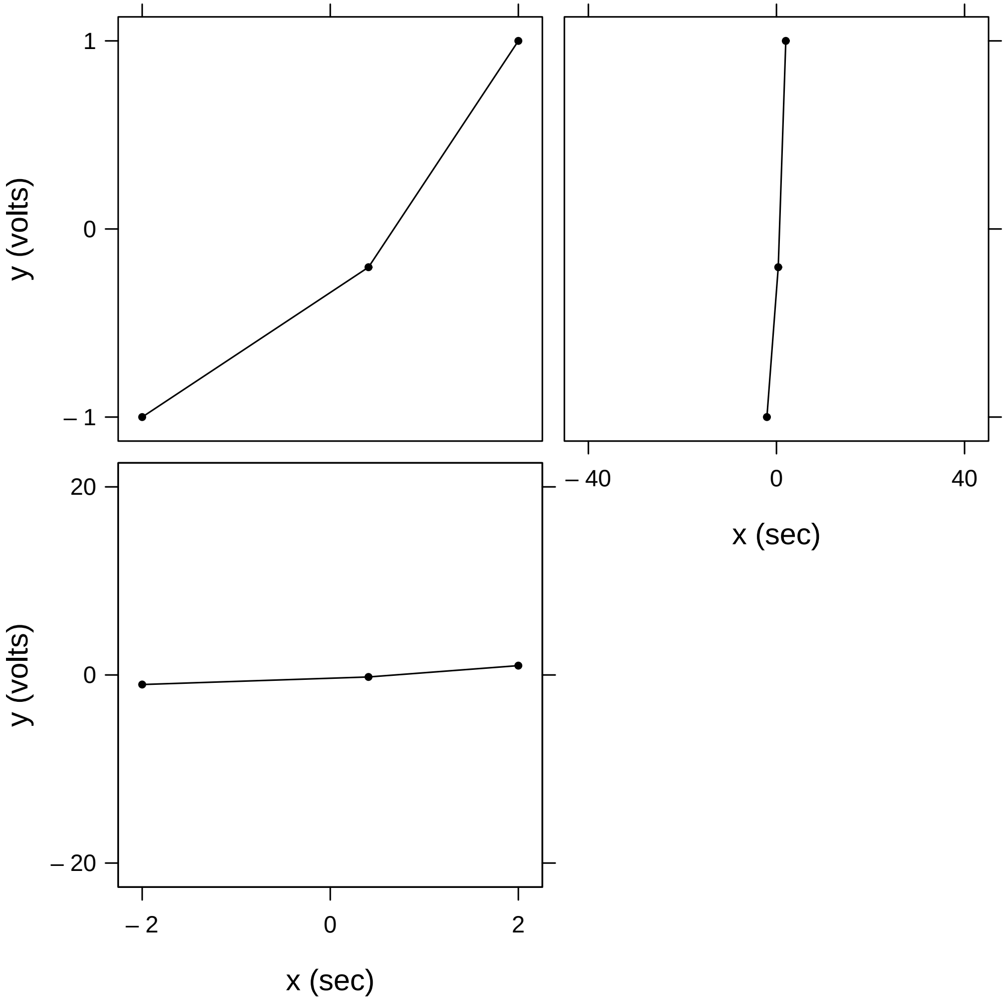


Banking to 45°

W. S. Cleveland and M. E. McGill and R. McGill, The Shape Parameter of a Two-Variable Graph, Journal of the American Statistical Association, 289-300, 83, 1988

A geometric observation and a designed experiment with subjects judging slope

Geometric Observation: Discrimination Orientation Differences



Visually discriminate difference in orientations of two segments with positive slopes

Best discrimination when the difference, d, of the orientations the largest

What value of a maximizes d?

$d \longrightarrow 0$ as $a \longrightarrow 0$ or $a \longrightarrow \infty$

d is a continuous function of a

Maximize Difference of Orientations

Let $a = \hat{a}$ be value of the aspect ratio that makes the average of two orientations equal to 45°

$$d(f) = \text{arctan}(f\hat{s}) - \text{arctan}(f/\hat{s})$$

Let $\hat{s}$ be the larger of the two physical slopes when $a = \hat{a}$

$$d'(f) = \frac{(1-f^2)(\hat{s} - 1/\hat{s})}{(1+f^2\hat{s}^2)(1+f^2/\hat{s}^2)}$$

Since average is 45°, each orientation is 90° minus the other, so the smaller physical slope is $1/\hat{s}$

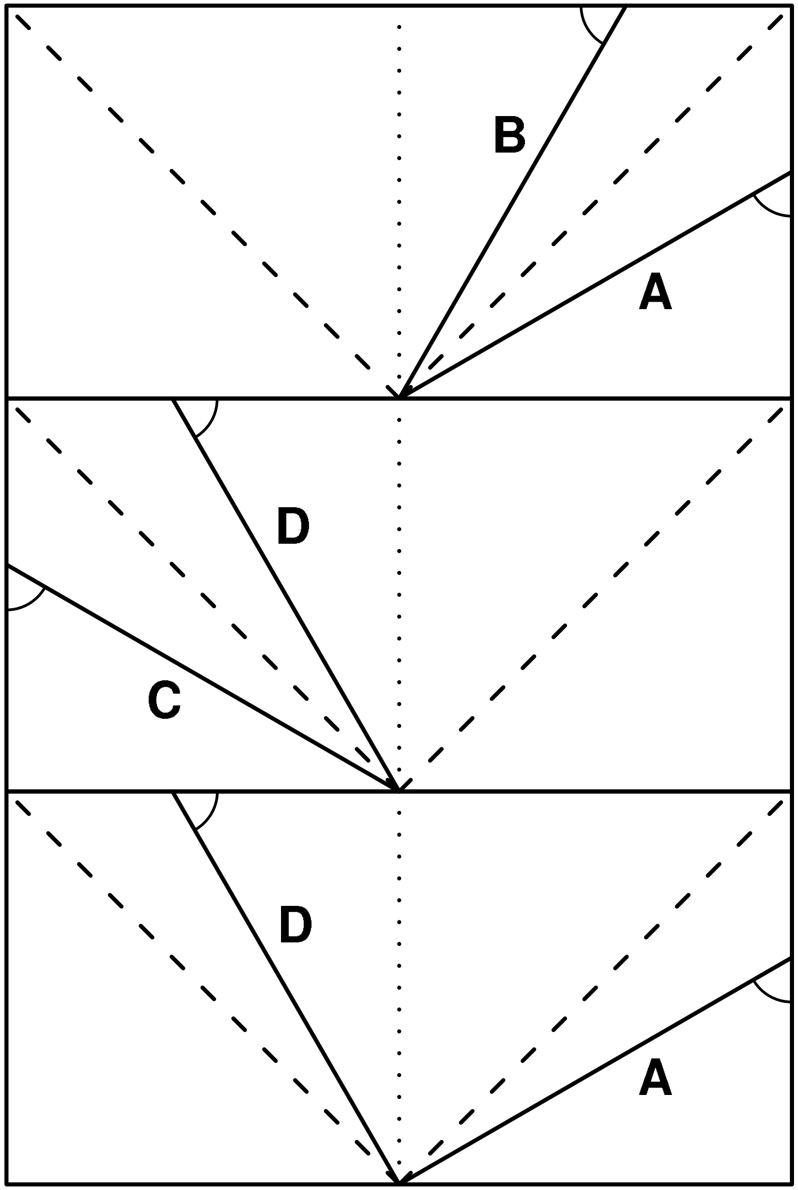
Let $a = f\hat{a}$ for $f > 0$

Two physical slopes are $f\hat{s}$ and $f/\hat{s}$

$$d'(f) = \begin{cases} \text{positive} & \text{if } f < 1 \\ 0 & \text{if } f = 0 \\ \text{negative} & f > 1 \end{cases}$$

Difference is maximized when a is chosen so that mean orientation is 45°

The Three Cases



Chose a to maximize differences of the absolute orientations of a pair of orientations

What this achieves for the three cases of pairs

- 1. Two positives: maximizes difference of orientations
- 2. Two negatives: maximizes difference of orientations
- 3. One negative and one positive: maximizes difference in the steepness

In the three cases, the segments make equal angles with the horizontal and the vertical

Banking to 45°

For a collection of segments, choose a to center $|\theta_k|$ on 45°

If we take this general principle and add to it a specific definition of the meaning of centering on 45°, then the result is a banking automation algorithm for the aspect ratio

Cleveland, McGill, and McGill suggested choosing a so that median absolute orientation is 45°

A number of other suggestions

(A vast literature not tied to slope judgement, beginning in 1910)

Banking to 45°

J. Heer and M. Agrawala, Multi-Scale Banking to 45 Degrees (2006), IEEE Transactions on Visualization and Computer Graphics, 12, 701–708

Choose a to maximize

$$\tau(a) = \sum_{j,k=1}^{n} (|\theta_j| - |\theta_k|)^2$$

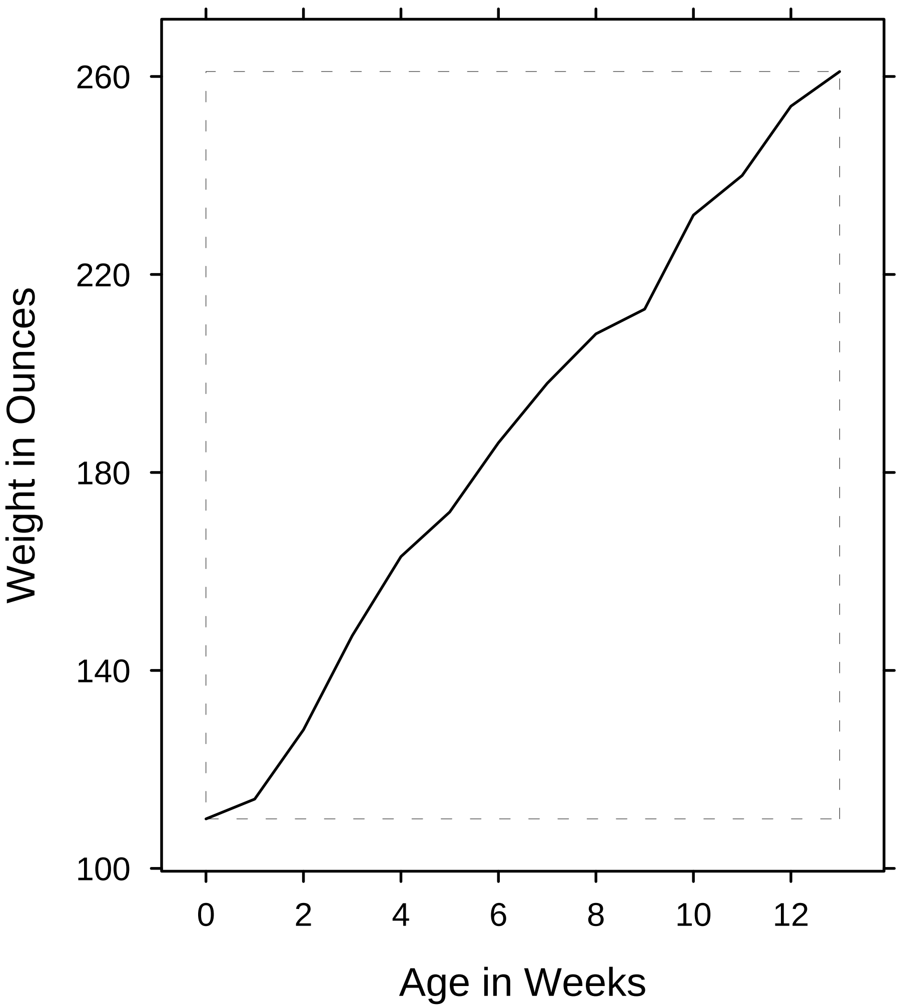
We have

$$\tau(a) = 2n \sum_{k=1}^{n} (\, |\theta_k| - \overline{|\theta|} \,)^2,$$

where $\overline{|\theta|}$ is the mean of the $|\theta_k|$, so their criterion maximizes the variance

R.A. Fisher: Statistical Methods for Research Workers, 1925

Graphed these data with $a = 1.18$ as shown here:



Founded both modern statistics and mathematical genetics

Very strong geometric intuition

Fisher writes, perceptively:
The features of such curves are best brought out if the scales of the two axes are so chosen that the graph makes with them approximately equal angles; with nearly vertical, or nearly horizontal lines, changes in the slope are not so readily perceived.

Fisher understood clearly this special case of a monotone curve. Did he understand the general case?

Studying the Properties of Banking Methods

How can we study banking methods?

Current methods in use do not have closed form solutions and must use numerical optimization

Properties have been studied empirically by trying methods on different collections of line segments

No theoretical study of mathematical and statistical properties, and only a small amount of attention to perceptual issues

Mostly engineering, little science

A Study of Mathematical, Statistical, & Perceptual Properties of Banking

1. A new banking method, *resultant-vector banking*
   - simple tractable algebraic formulas
   - geometrically motivated, which provides geometric intuition
   - canonical segment plot shows banked segments in an informative way

2. Going from the discrete case of n segments to differentiable case provides mathematical insight

3. Theory of curvature perception from the field of visual perception adds important perceptual insights

Visualization Generally Needs More Math, Stat, and Visual Perception

Now mostly engineering and little science

How do we start?

1. Keep the engineering going since it is doing a lot of good

2. Study the properties of our inventions using math, stat, and visual perception and see if we want to modify them

3. This runs in parallel with the most powerful approach to data visualization of all: developing, implementing, and using our inventions in our own data analyses

Resultant-Vector Banking to $45°$: Concepts

Starting point: the segments on a graph with aspect ratio a

Allowable changes, conceptually

- multiply slopes collectively by a factor f by changing the aspect ratio by factor f
- translate segments
- for a segment with a negative orientation $-\theta$, change orientation to θ, keeping length the same

Resultant-Vector Banking to 45°: Definition

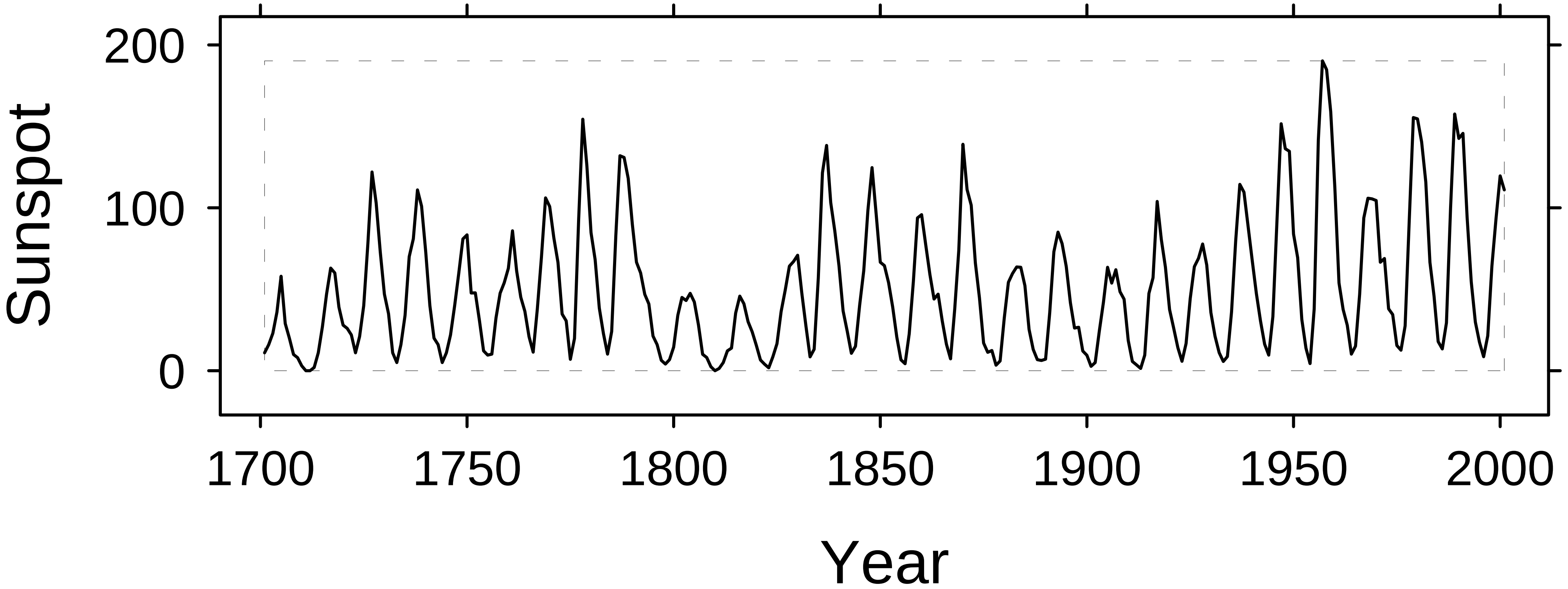
Start with a graph and change segments

1. Change signs of negative orientations

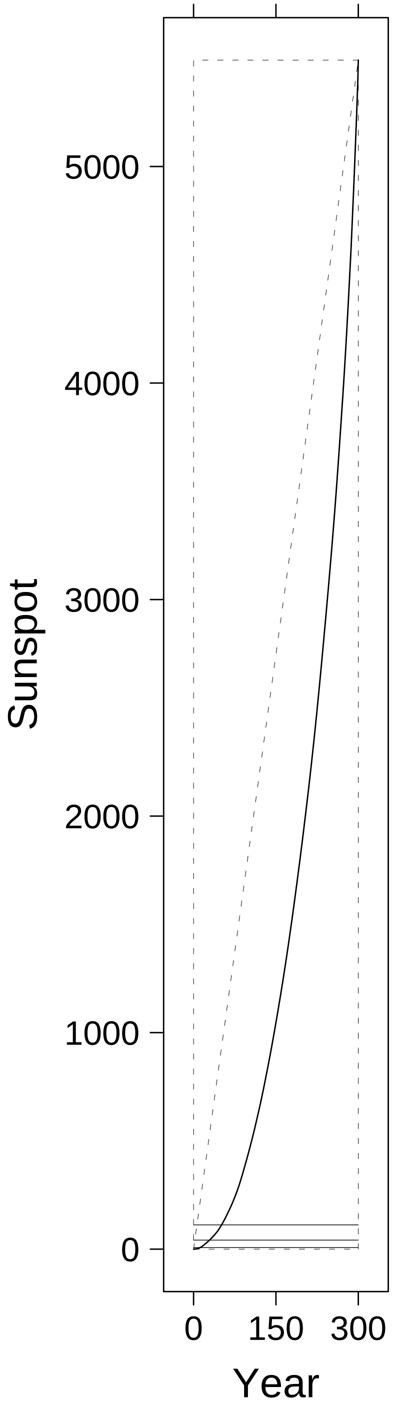
2. Translate them to form a resultant vector
   - segments are ordered by smallest to largest slope left right
   - (ordering not necessary but results in an informative display)
   - this is the canonical segment plot of the original display

3. Change the aspect ratio to give the resultant vector a slope of 1
   - this is the canonical segment plot of what will be the banked display

Original Display

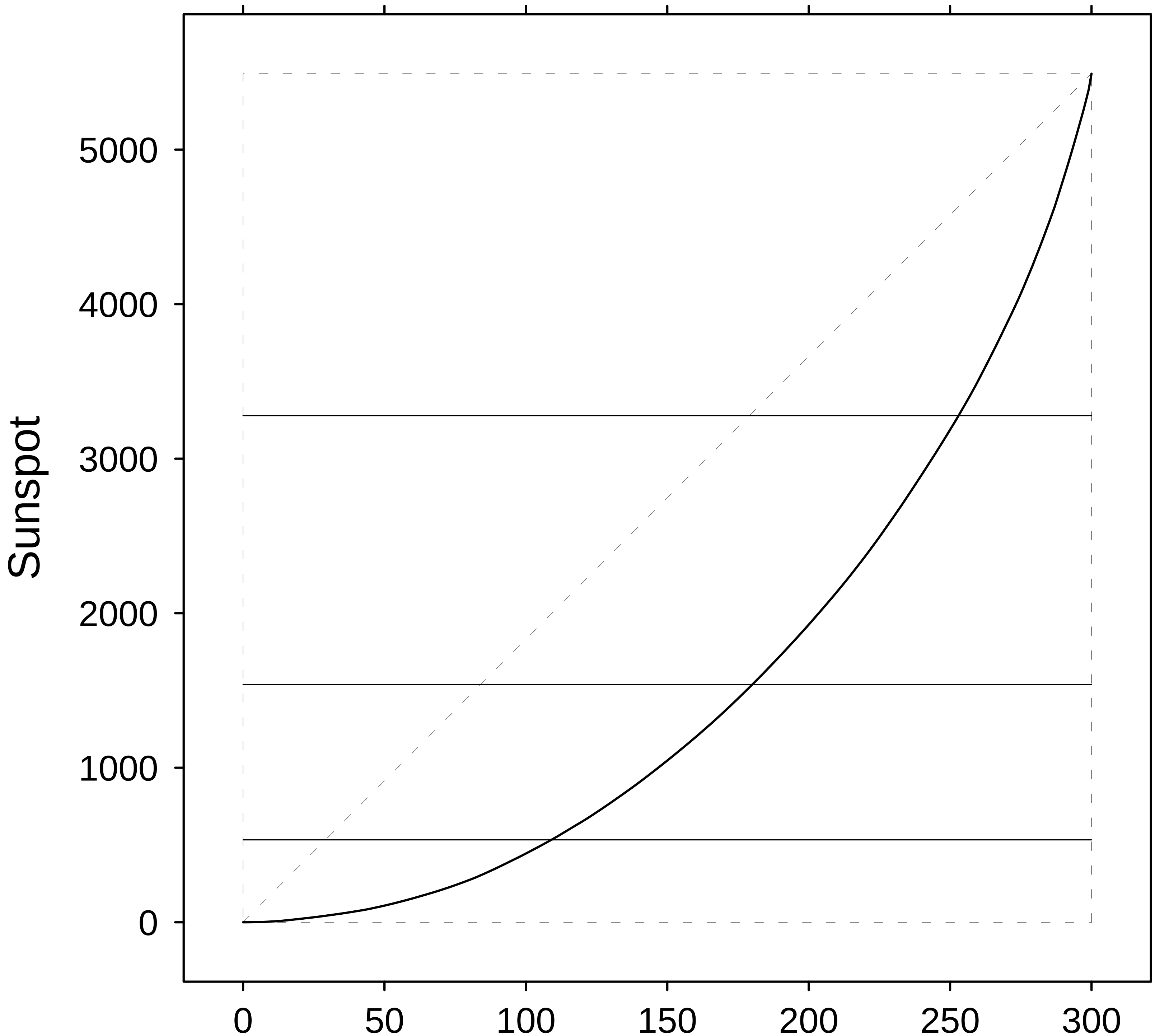


Canonical Segment Plot: Original Display

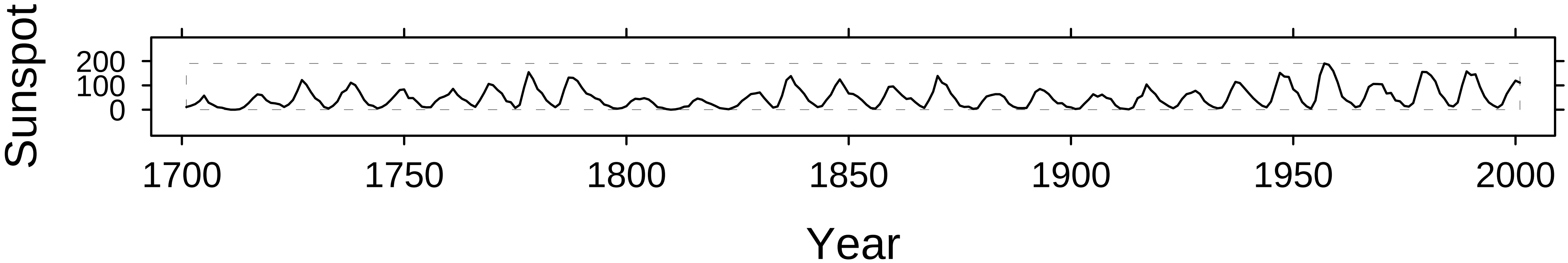


Three horizontal lines show
$30°, 45°, 60°$

Canonical Segment Plot: Banked Display



Banked Original Display



Orientations on banked canonical segment plot are those on banked original but with minus signs restored to those that were originally negative

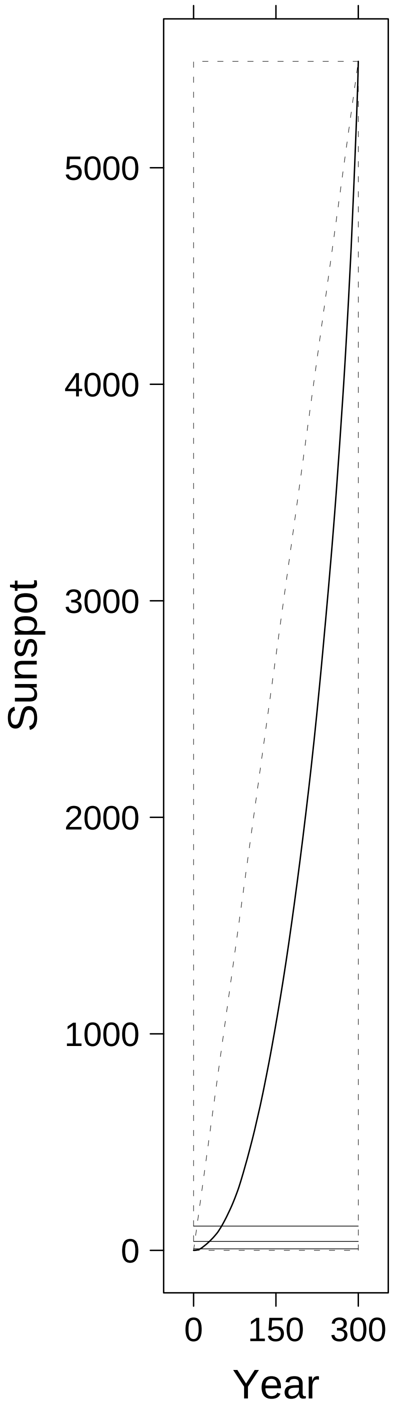
a = aspect ratio of original display

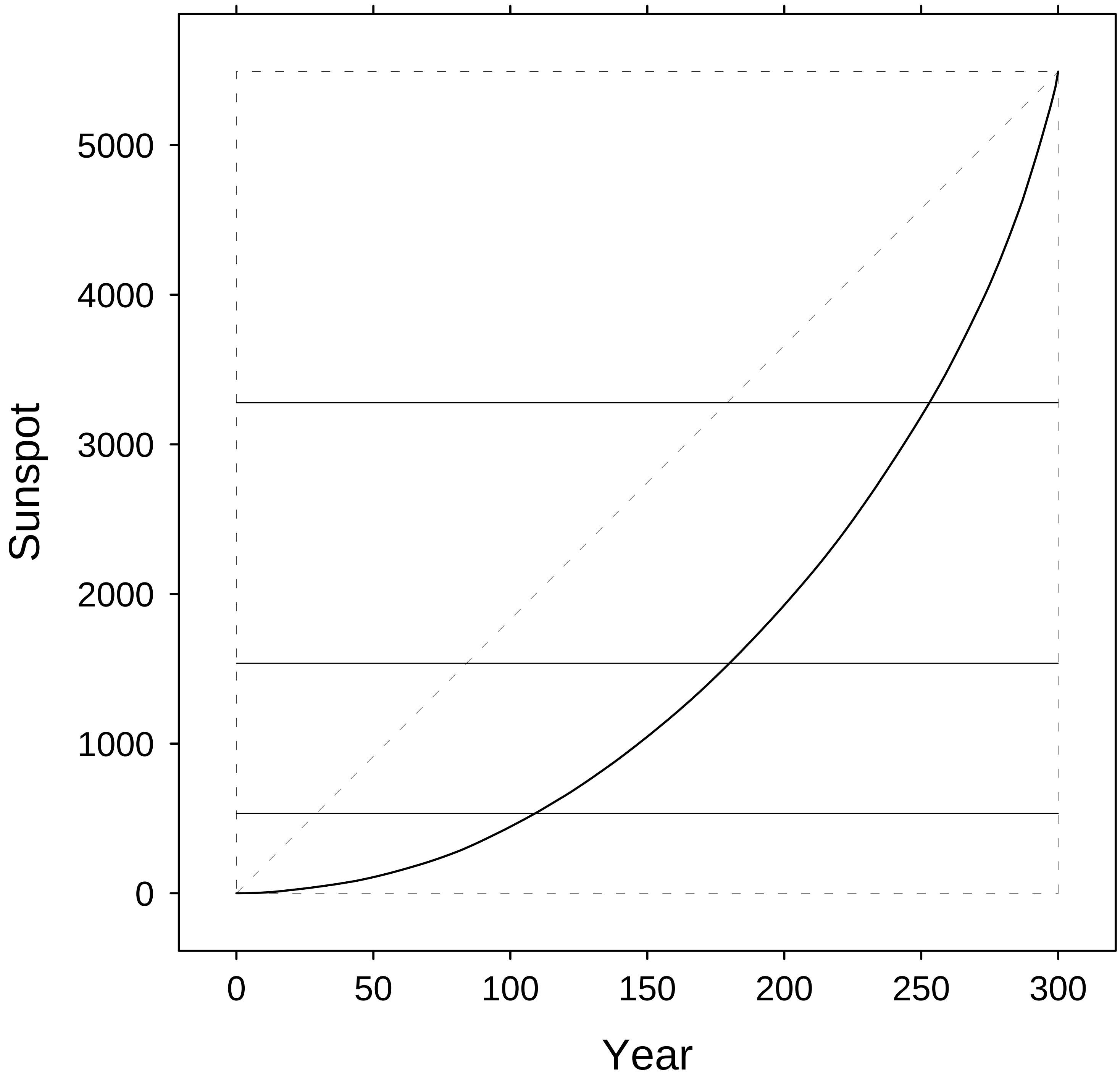
a^* = aspect ratio of canonical segment plot of original display = slope of resultant

$\hat{a}^* = 1$ = aspect ratio of banked canonical segment plot

$\hat{a}$ = aspect ratio of banked original = a/a^*

Two Canonical Segment Displays





A Simple Formula

n = number of segments

$x_k = (x_{1k}, x_{2k})$
physical horizontal values of the left and right endpoints of segment k
$x_{1k} < x_{2k}$, and x_{1k} is nondecreasing in k

$y_k = (y_{1k}, y_{2k})$
physical vertical values of segment endpoints

$\Delta x_k = x_{2k} - x_{1k} \quad \Delta y_k = y_{2k} - y_{1k}$

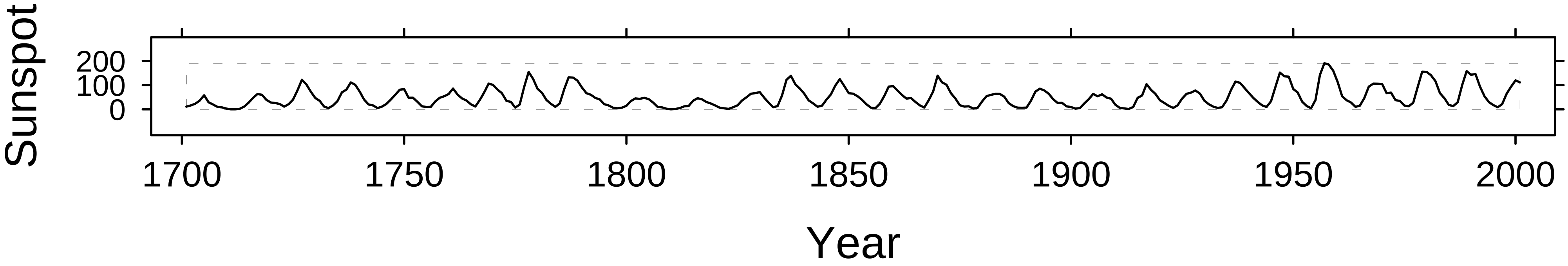
Range:
$R_x = \mathrm{range}_{k=1}^{n}(x_k) \qquad R_y = \mathrm{range}_{k=1}^{n}(y_k)$

Variation:
$V_x = \sum_{k=1}^{n} \Delta x_k \qquad V_y = \sum_{k=1}^{n} |\Delta y_k|$

$$\hat{a} = \frac{R_y / V_y}{R_x / V_x}$$

Formula also true with physical units of x_k and y_k changed to data units

An Even Simpler Formula for a Connected Segment Plot



For connected segment plot

$$x_{2k} = x_{1(k+1)}, k = 1, \ldots, n-1$$

$$\mathrm{range}_{k=1}^{n}(x_k) = \sum_{k=1}^{n} \Delta x_k$$

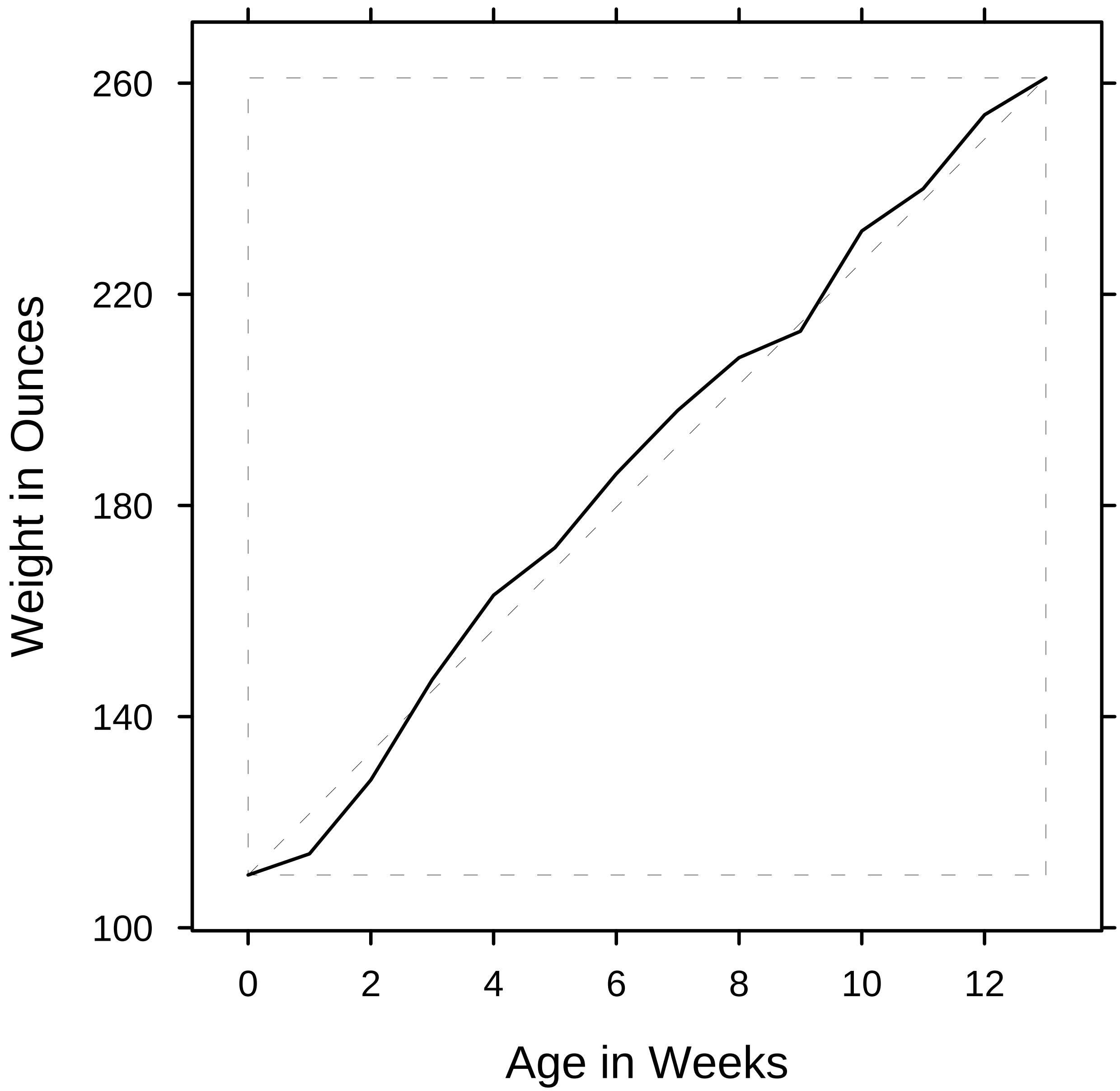
$$R_x = V_x$$

$$\hat{a} = \frac{R_y}{V_y}$$

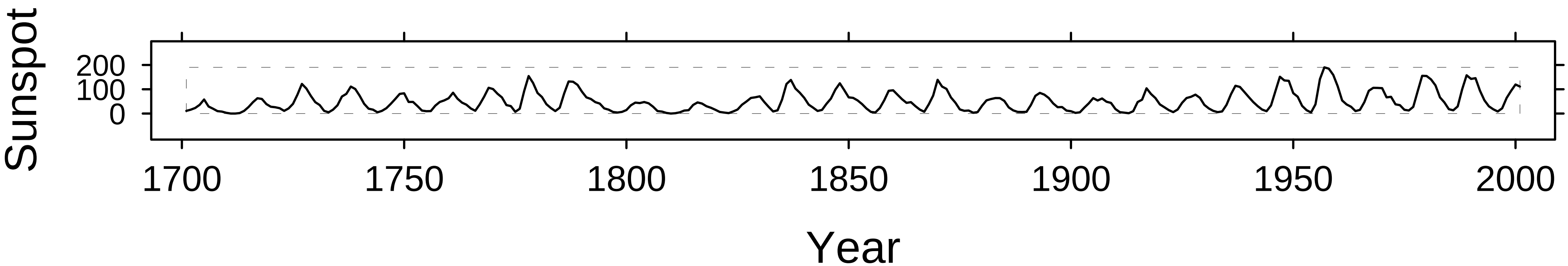
Formula also true with physical units of y_k changed to data units

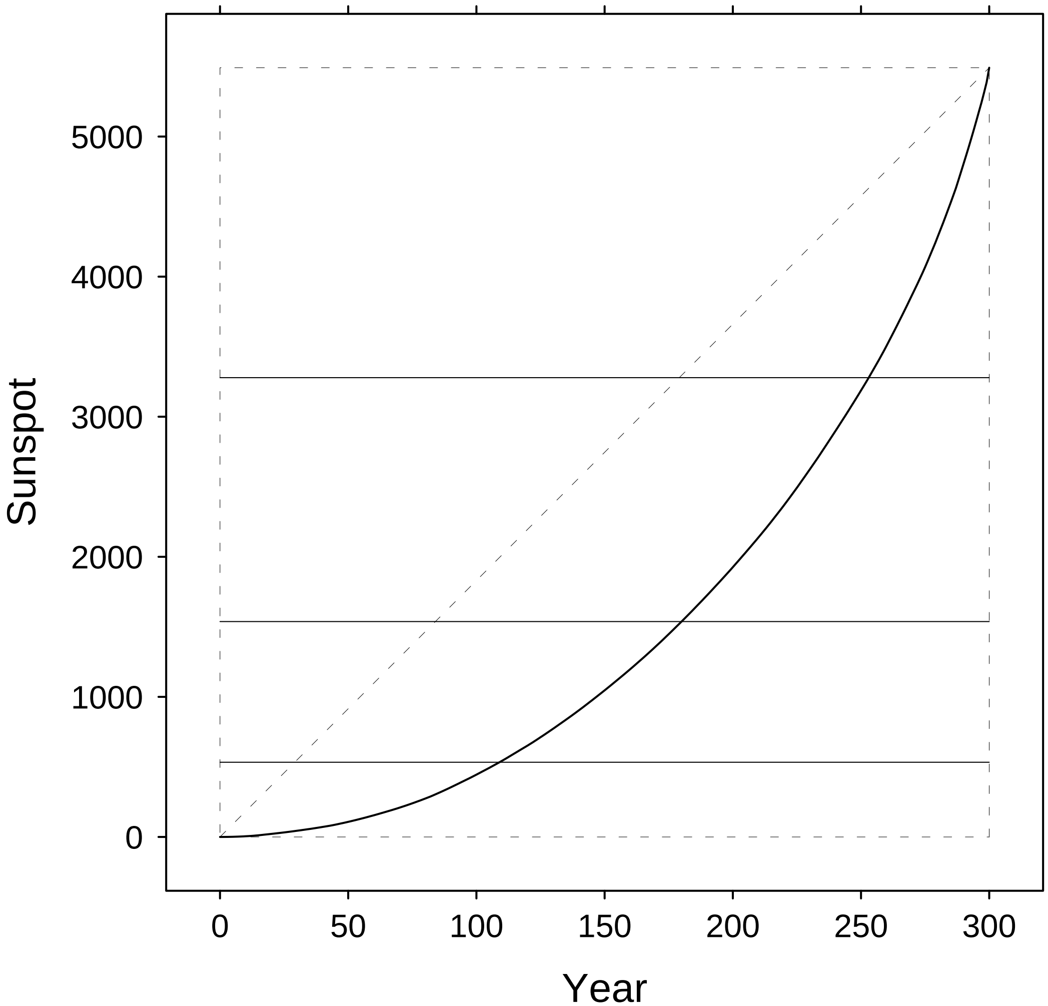
Connected Segment Plot: Properties from Geometry

$\hat{a} = 1$ for monotone nondecreasing segments. Segments of the original display form a resultant vector.
$\hat{a} = 1$ for monotone nonincreasing segments. Similar reasoning.



Connected Segment Plot: Properties from Geometry





$a = \hat{a} \leq 1$ for a banked display

Translations and sign changes of segments to construct canonical segment display increases a or leaves it the same and $\hat{a}^* = 1$

Geometry and Algebra

Results from the geometry can be derived from the formula too

The geometry yields substantial intuition in seeking the truths of things we do not understand

The formulas can provide simpler proofs in some cases and are needed to study statistical properties

Connected Segments Plot for Statistical Time Series

Time series y_t for t = 1 to n plotted against t by connected segment plot

$y_t = \epsilon_t$ for t = 1 to n where ϵ_t is gaussian white noise: i.i.d N(0,σ^2)

Convergence in probability of $\hat{a}(n)$ as $n \longrightarrow \infty$

$\hat{a} \longrightarrow 0$

$y_t = t^p + \epsilon_t$ where $p > 0$, trend plus noise. $\hat{a} = 1$ for k^p plotted alone but ϵ_t pushes toward 0

For $p > 1$, $\hat{a} \longrightarrow 1$. For $p < 1$, $\hat{a} \longrightarrow 0$.

For $p = 1$, there is a compromise. $\hat{a} \longrightarrow \hat{a}(\infty)$.

$$\hat{a}(\infty) = \frac{2\sigma \exp(-1/4\sigma^2)}{\sqrt{\pi}} + 2\Phi(\frac{1}{\sigma\sqrt{2}}) - 1,$$

where Φ is the N(0,1) distribution function. For σ very close to 0, the white noise has small variability and $\hat{a}_\rho(\infty)$ is close to 1. As σ increases, the noise variability increases, and $\hat{a}(\infty)$ decreases monotonically to 0.

Mathematical Functions: Segments to Derivatives

Suppose $y(x)$ is function on $[\alpha, \beta]$ with a bounded continuous derivative

Variation:

$$V = \int_{\alpha}^{\beta} |y'(x)| dx$$

Range:

$$R = (\mathrm{Range}_{\alpha}^{\beta}(\mathrm{y}))$$

Banking aspect ratio:

$$\hat{a} = \frac{V}{R}$$

Our geometric intuition does not change at all

Minmax Partition for Piecewise Linear Approximation

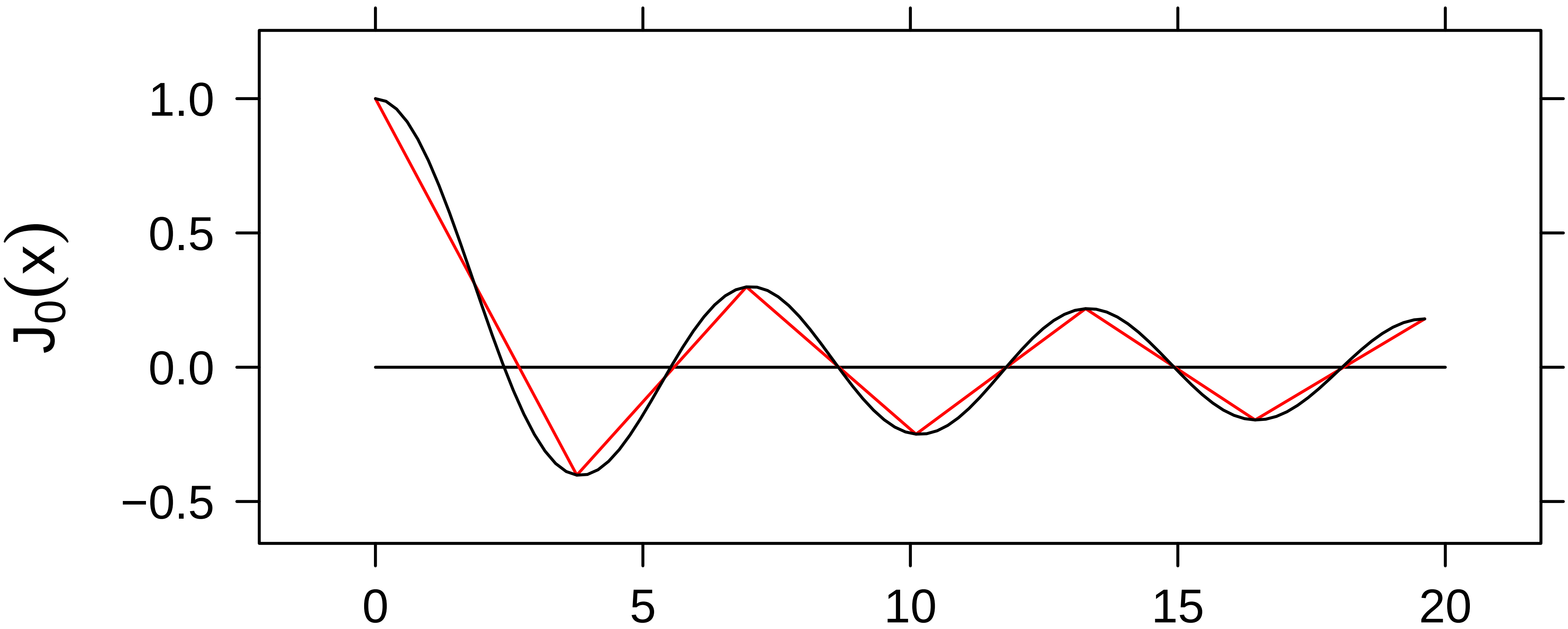
Choose x_k to be α, β, and values at which minima and maxima occur

Function is monotone between successive partition points

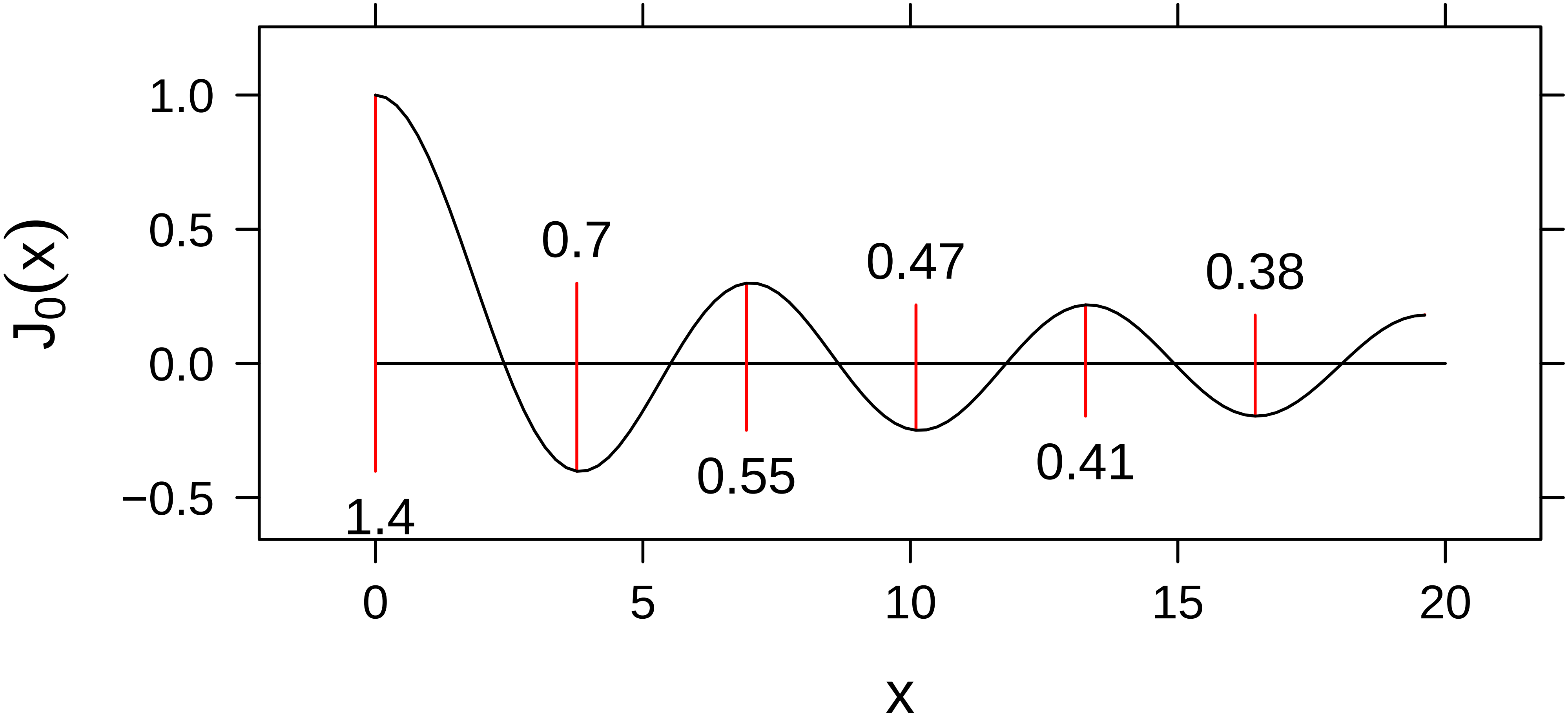
Variation between two success points is absolute difference of y at the points

We can take the piecewise linear approximation and use its segments to carry out resultant-vector banking. Each linear piece is the resultant vector for interval

Example shows a bessel function



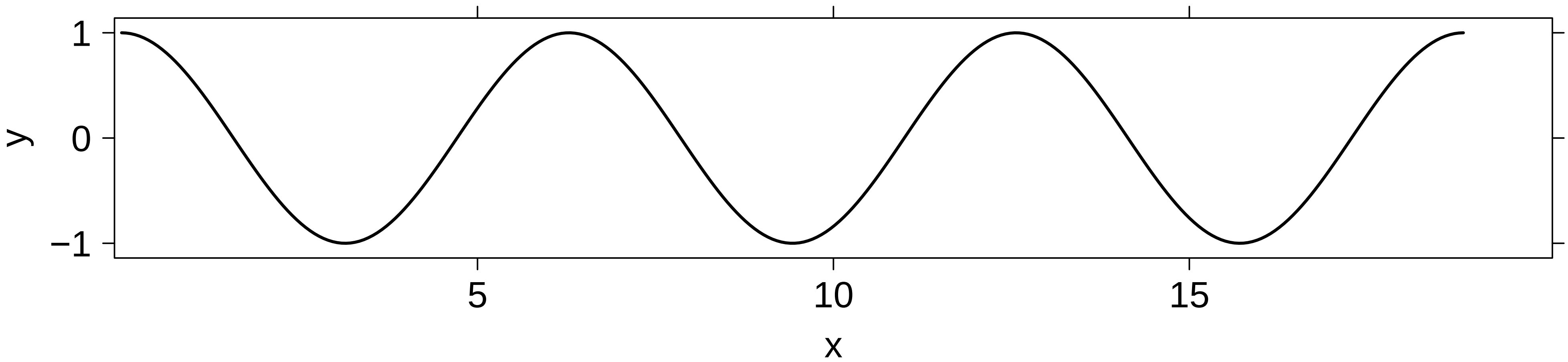
Minmax Partition for Piecewise Linear Approximation



$$\hat{a} = \frac{1.4}{1.4 + 0.7 + 0.55 + 0.74 + 0.41 + 0.38} = 0.357$$

Each monotone piece wants its own data rectangle to have an $\hat{a}$ of 1; global $\hat{a}$ is a compromise

Another Example: Cosinusoid



$\cos(2\pi m x)$ for positive integer m: goes through m cycles (m = 3 above)

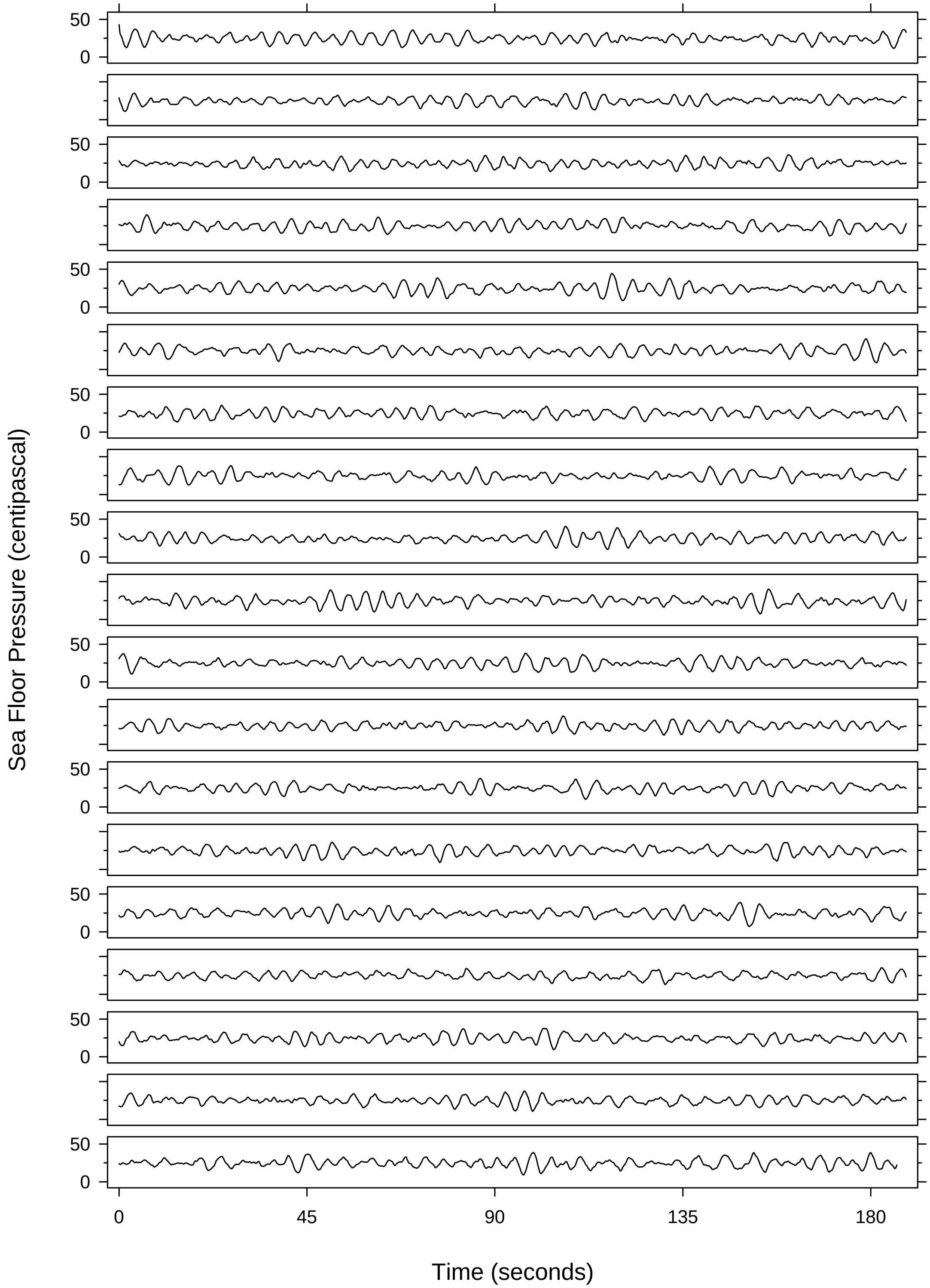
$V = 2m$: $2m$ monotone intervals, each with variation 2

$R = 2$

$\hat{a} = V/R = 1/2m$, so $\hat{a} \to 0$ $m \to \infty$

Each monotone interval gets its own desirable $a = 1$

Quasi-Periodic Time Series



Planar Curvature Perception

The planar curve $(x(t), y(t))$

In visual perception, a model for the curvature that we perceive in a planar curve

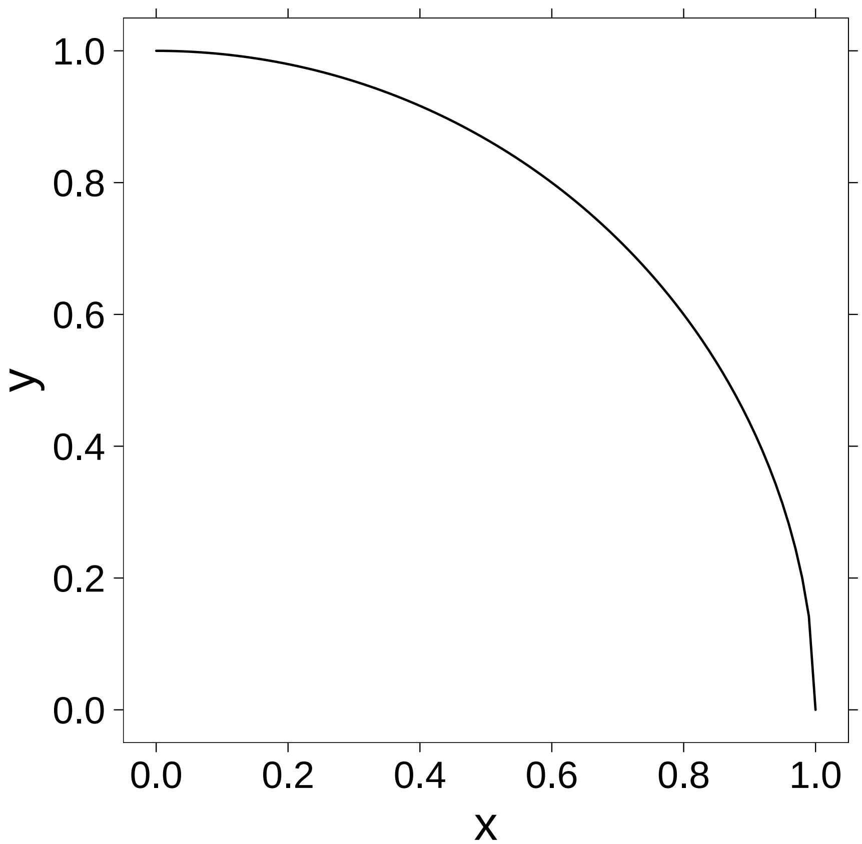
- derivative of the orientation of the tangent w.r.t. curve length
- perception is effortless and the curve forms a gestalt

If the curve is specified as a function $y(x)$ of x, then derivative of $\arctan(y'(x))$ is w.r.t $\sqrt{y^2 + x^2}$ not x

This gestalt curvature is

$$C_g(x) = \frac{y''(x)}{(1 + f'^2(x))^{3/2}}$$

A Circular Function



$$y(x) = \sqrt{1 - x^2}$$

$$y'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

$$y''(x) = \frac{-1}{(1 - x^2)^{3/2}}$$

$$C_g(x) = -1$$

We see the curve as an object with a circular form

But we can separately assess the 1st and 2nd derivatives

Perception of Mathematical Dependence

The goal in displaying a function typically is to gain an understanding of how y depends on x

Formation of a curve gestalt and assessing functional dependence are different visual processes

Our visual perceptions for function understanding are guided by cognitive interest in the steepness of a curve at x and how it changes with x

What we see for assessing steepness is likely more accurated served by the derivative of the orientation of the tangent w.r.t x:

$$C_f(x) = [\arctan(y'(x))]' == \frac{y''(x)}{1 + y'^2(x)}$$

For the circular function

$$C_f(x) = \frac{-1}{\sqrt{1 - x^2}}$$

C_f and Banking to $45°$

Suppose $y'(x)$ are the physical slopes of the function on a graph when $a = 1$

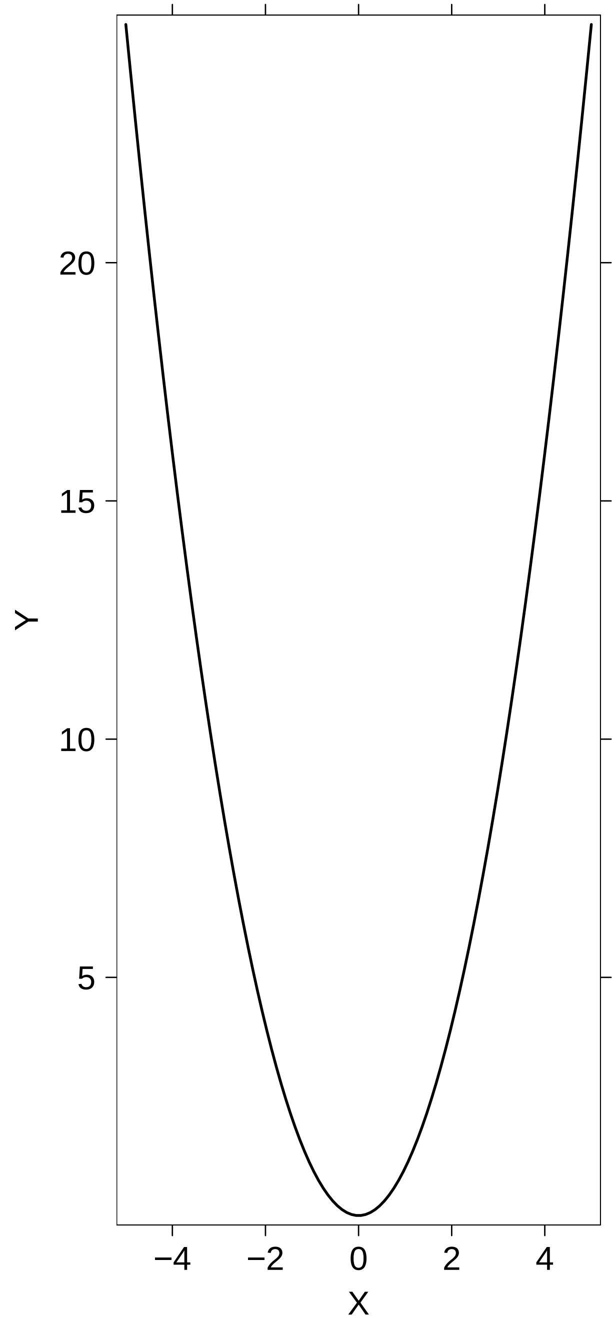
For aspect ratio a, the physical slopes are $ay'(x)$ and

$$C_f(x) == \frac{ay''(x)}{1 + a^2 y'^2(x)}$$

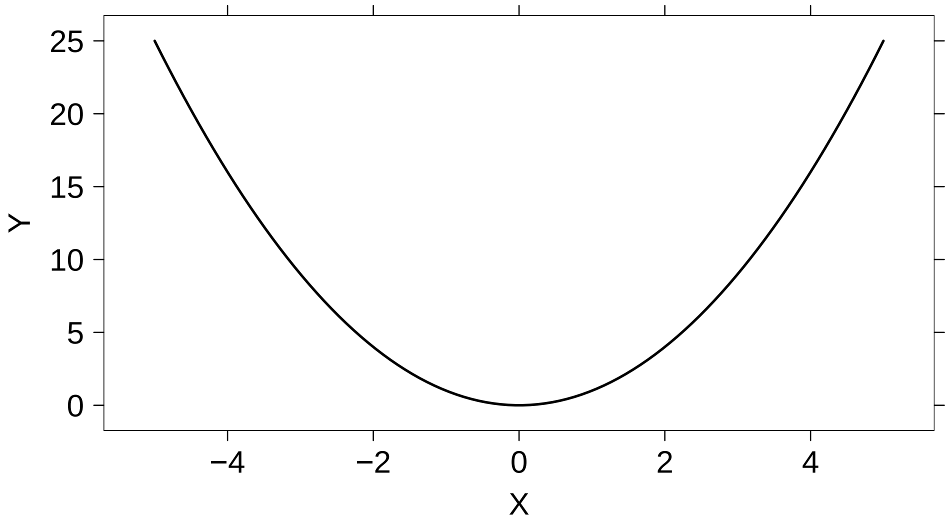
The value of a that maximizes $|C_f|$ at x is $a = 1/|y'(x)|$, so for this a, the physical slope at x is $\pm 45°$

This agrees with the discrete segment case

The Vanishing 2nd Derivative of the Quadratic



Banking Helps But is Not a Complete Cure



We get insight from C_f for the quadratic

$$C_f = \frac{2}{1 + 4x^2}$$

What we perceive is a rate of change of the orientation of the tangent that goes to 0 as x increases