PDE Transform for signal, image and data analysis

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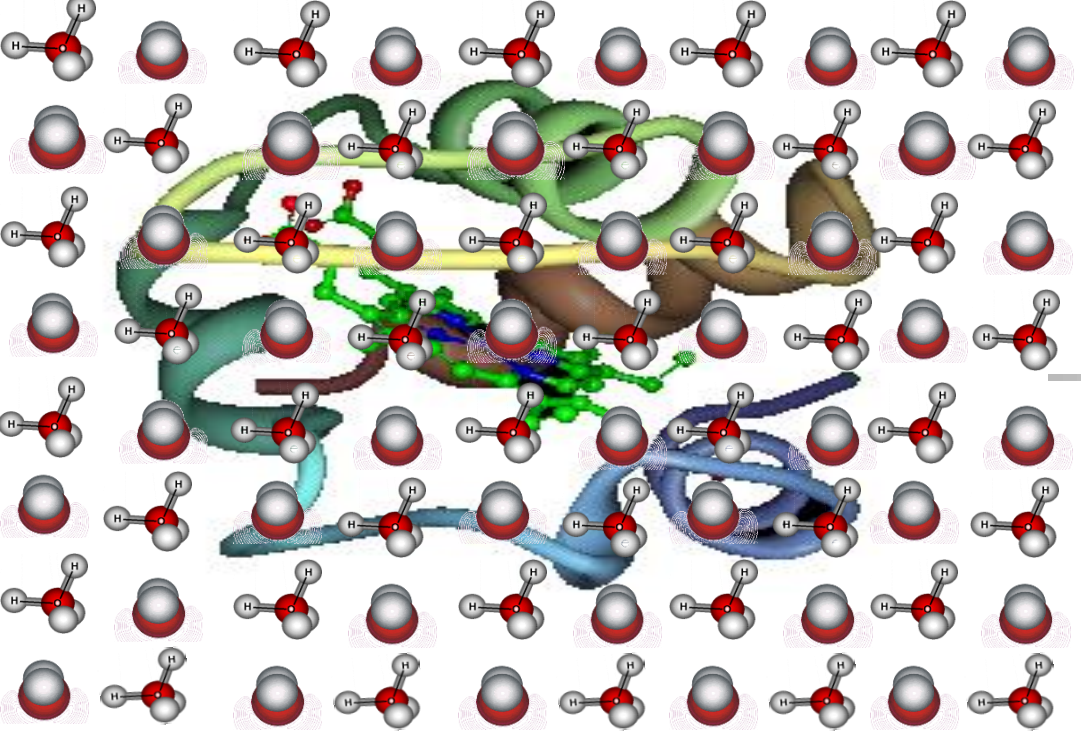
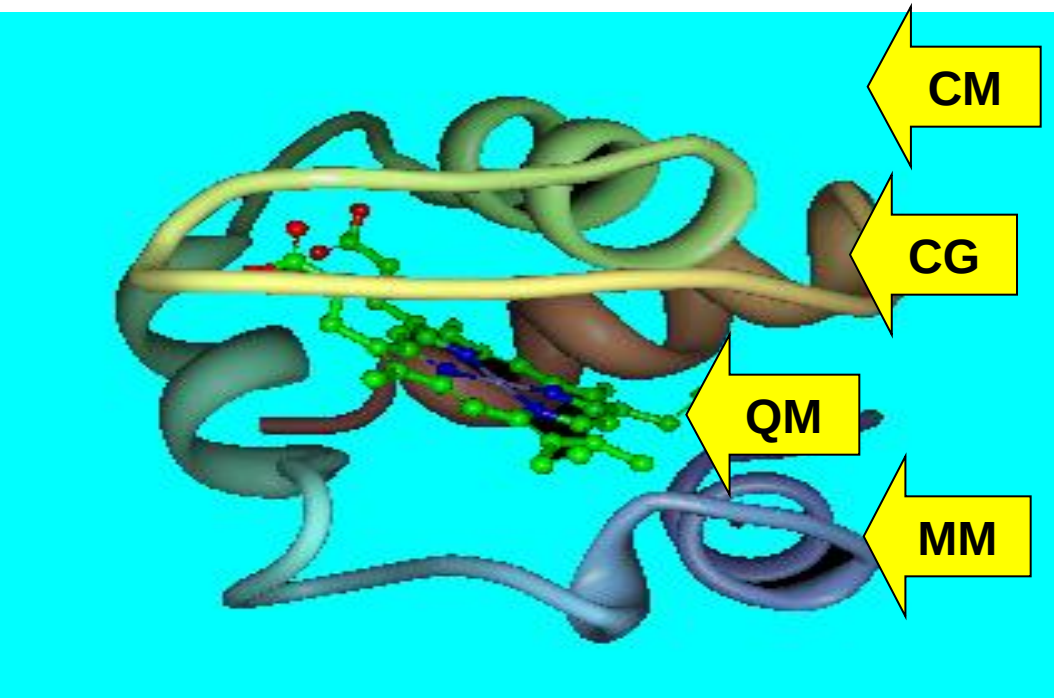
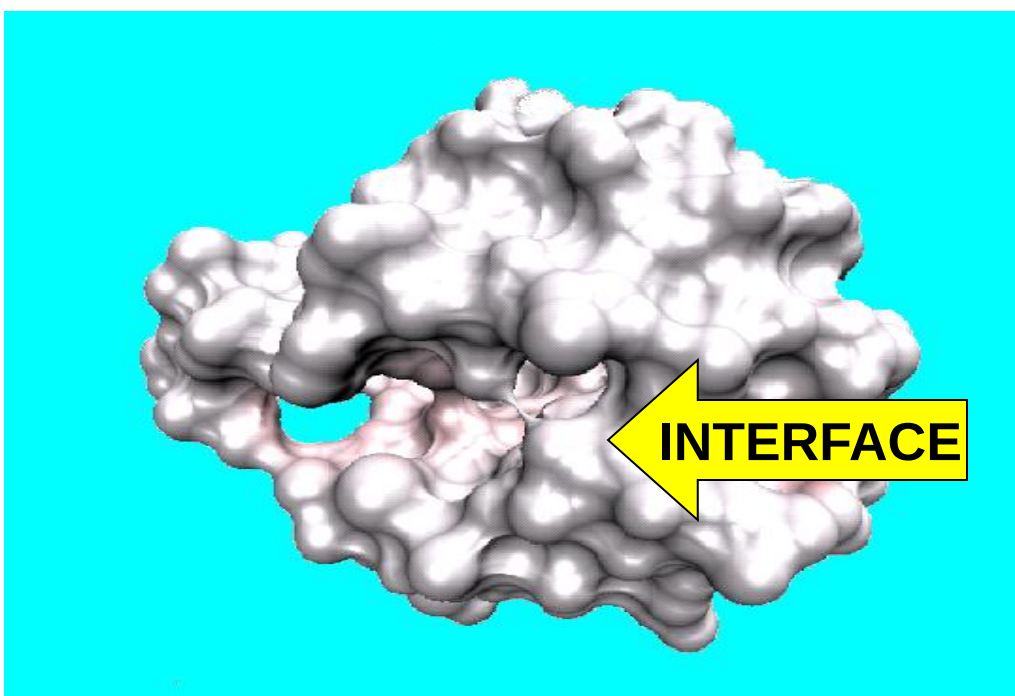


Keyword Summary:

- *Variational Multiscale/Differential geometry*
- *Continuum/Hydrodynamics*
- *Discrete/Molecular dynamics/QM*
- *Coarse graining /Stable manifold /Machine learning*
- *22 reviewed journal papers*





Diffusion equation for image processing
(Withkin 1983)

$$u_t(r,t) = d\nabla^2 u(r,t)$$
$$u(r,0) = I(r)$$

Constant

Scale-space filter
Gaussian filter

Original

Original+Noise

Processed with the diffusion equation

Perona-Malik equation (1990)

$$\frac{\partial u}{\partial t} = \nabla \bullet \left[d\left(|\nabla u|\right) \nabla u \right],$$

$$u(r, t = 0) = I(r)$$

Gradient dependent

$$d\left(|\nabla u|\right) = \exp\left(-\frac{|\nabla u|^2}{2\sigma^2} \right), \qquad or$$

$$d\left(|\nabla u|\right) = \frac{1}{1 + |\nabla u|^2}$$

Mean Curvature Flow

Original

The First high-order stochastic geometric PDEs introduced for image analysis

Wei (1999); Greer & Bertozzi (2004); Gilboa, Sochen & Zeevi (2004); Xu and Zhou (2007); ….

$$\frac{\partial u}{\partial t} = \sum_{j=0} \nabla \bullet \left[d_j \left(|\nabla u| \right) \nabla \nabla^{2j} u \right] + P\left(|\nabla u| \right),$$

$$u(r, t = 0) = I(r)$$

$$d_0\left(|\nabla u| \right) = \exp\left(-\frac{|\nabla u|^2}{2\sigma^2} \right),$$

$$\sigma^2 = \overline{\left| \nabla u - \overline{\nabla u} \right|^2}$$

Gradient dependent

Nonlinear term

Stochastic coef.

Partial differential equation (PDE) transform

$$\frac{\partial u}{\partial t} = \sum_{i=0}^{n-1} \nabla \bullet \left[d_i \left(|\nabla u| \right) \nabla \nabla^{2i} u \right] + c_u (v - u),$$

$$\frac{\partial v}{\partial t} = \sum_{j=0}^{m-1} \nabla \bullet \left[d_j \left(|\nabla v| \right) \nabla \nabla^{2j} u \right] + c_v (u - v)$$

Wang et al, IJNMBE 2011

$$u(r, t = 0) = v(r, t = 0) = I(r)$$

Intrinsic mode functions (IMFs)

$$w_{nm}^k = u_n - v_m = H_{nm} X^k, \qquad \forall k = 1,2,...$$

$$X_{nm}^1 = I(r)$$

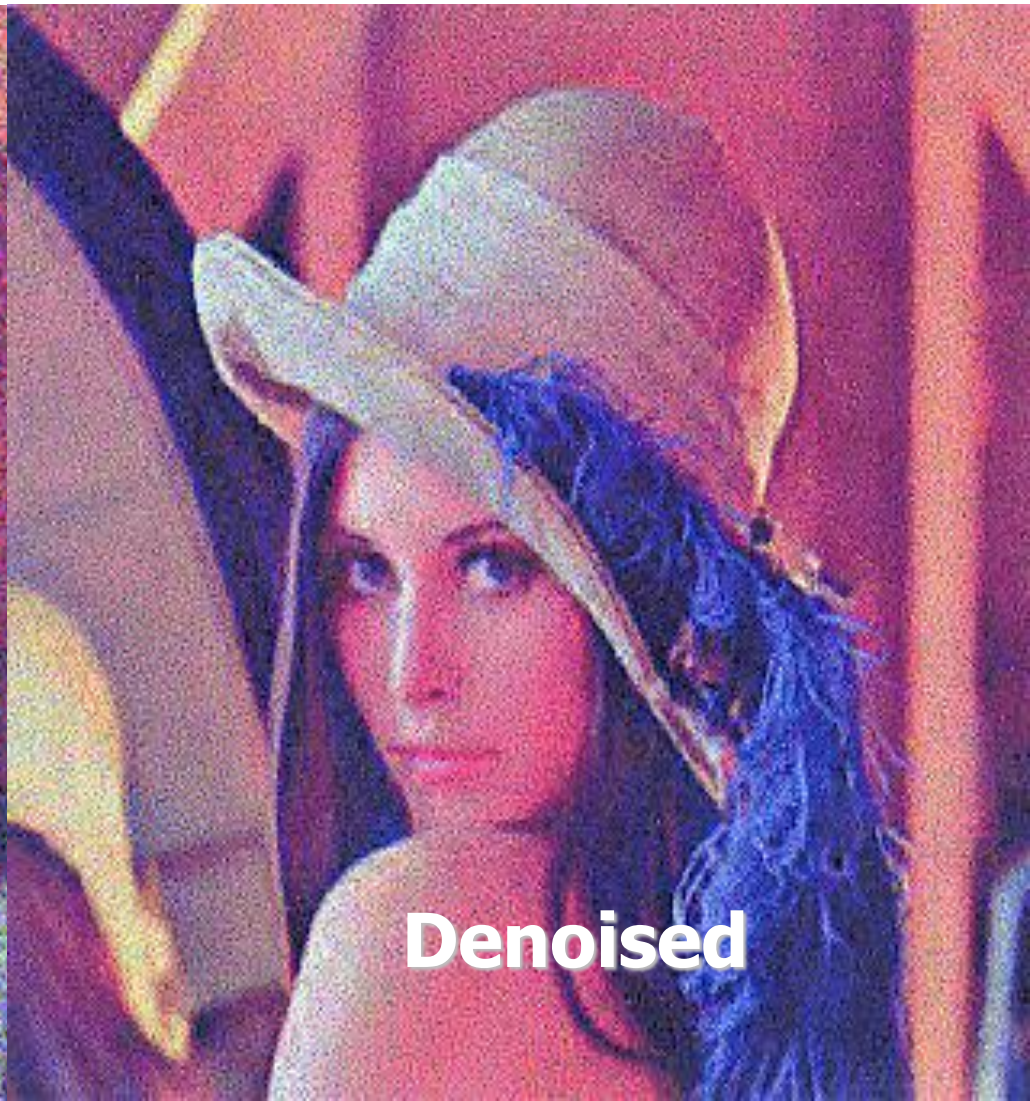
$$X_{nm}^k = X_{nm}^1 - \sum_{l=1}^{k-1} w_{nm}^l, \qquad \forall k = 2,3,...$$

$$I = X_{nm}^k + \sum_{l=1}^{k-1} w_{nm}^l$$

Perfect reconstruction

Use of Cahn-Hilliard type of potential

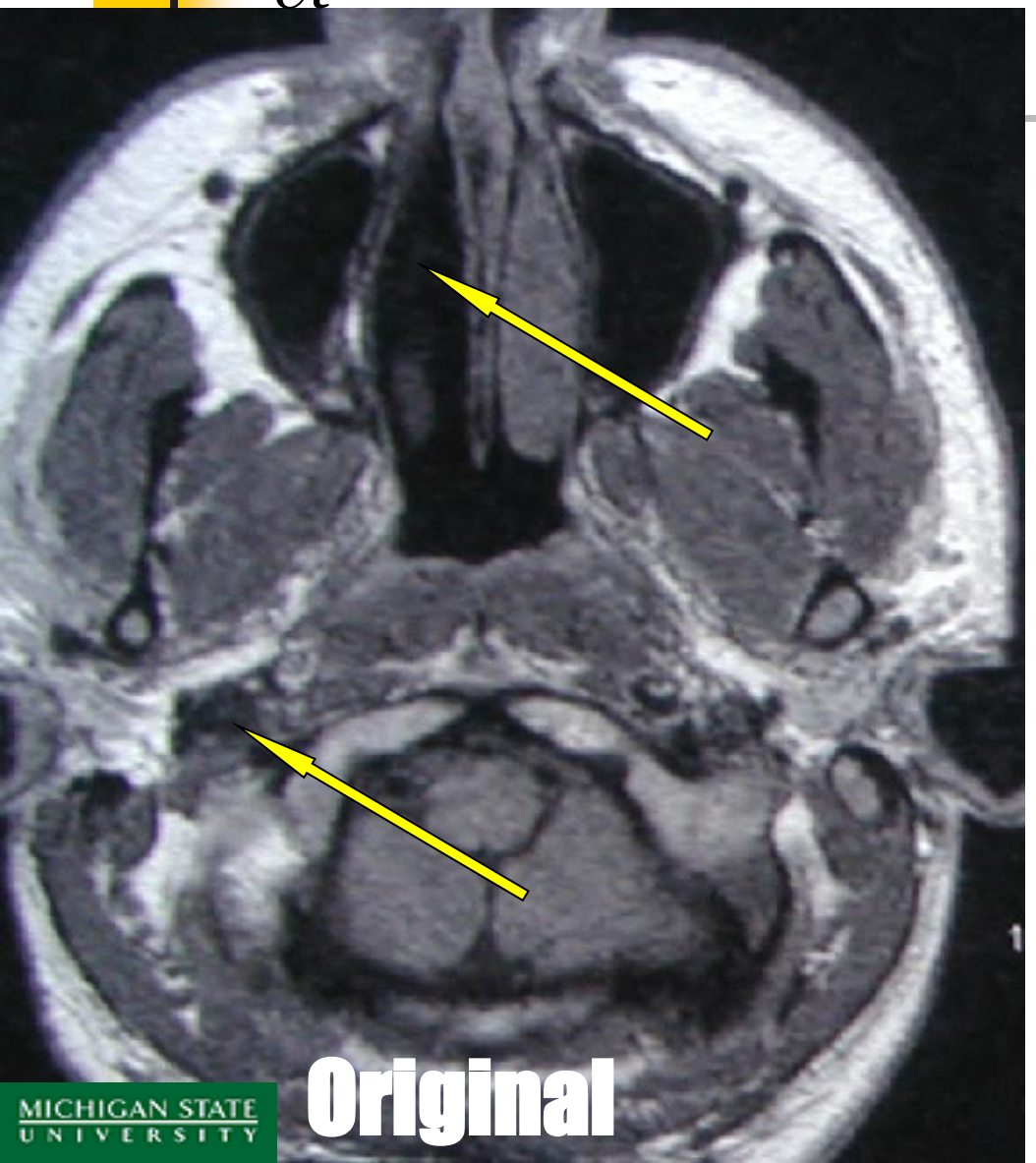
$$\frac{\partial u}{\partial t} = \nabla \bullet \left[d_1(|\nabla u|) \nabla \nabla^2 u \right] + c(|\nabla u|)(u^2 - u_0^2)u$$



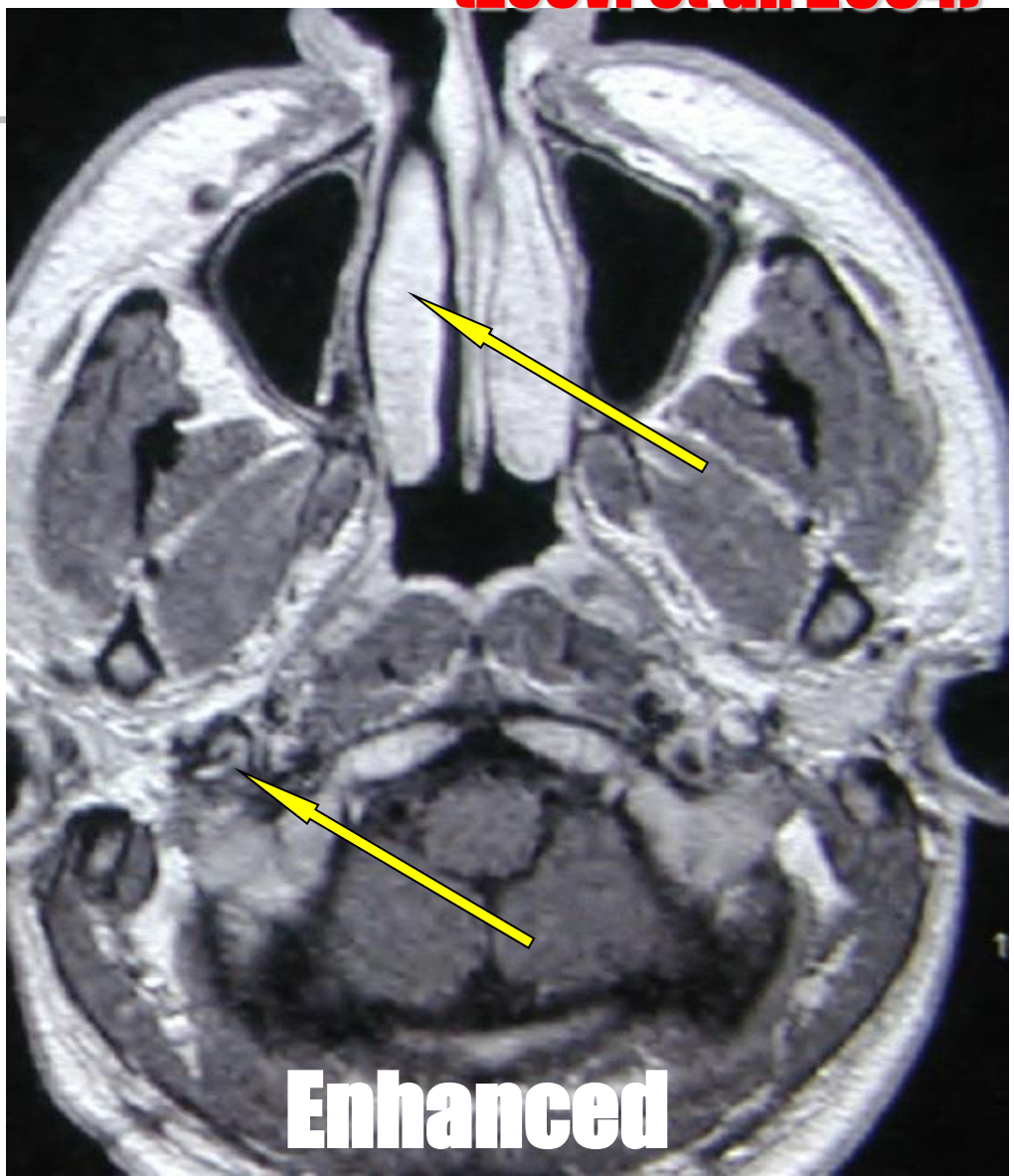
Forward & Backward enhancement

$$\frac{\partial u}{\partial t} = \nabla \bullet \left[d_0 \left(|\nabla u| \right) \nabla u \right] + \nabla \bullet \left[d_1 \left(|\nabla u| \right) \nabla \nabla^2 u \right]$$

(Zeevi et al. 2004)

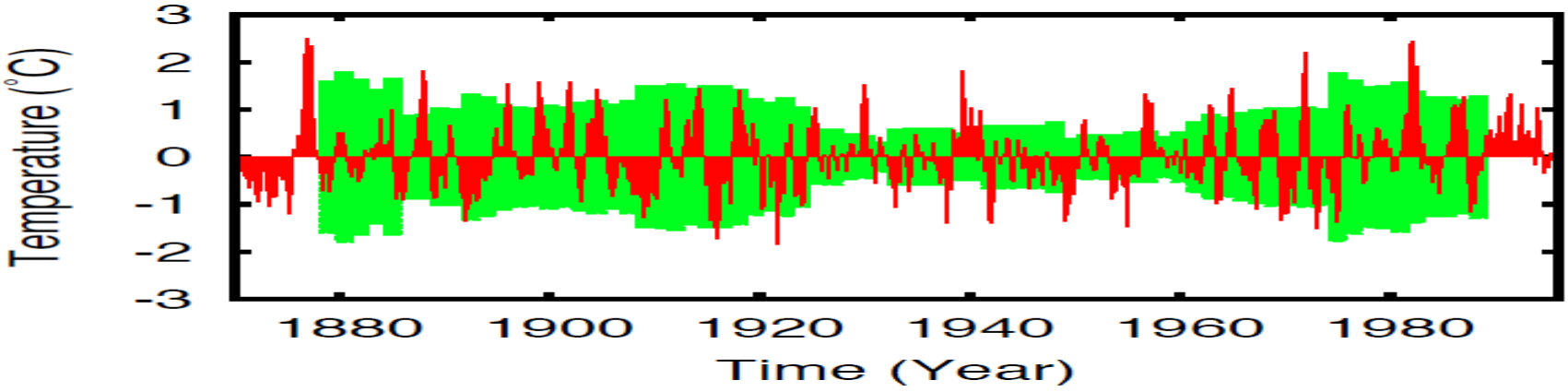


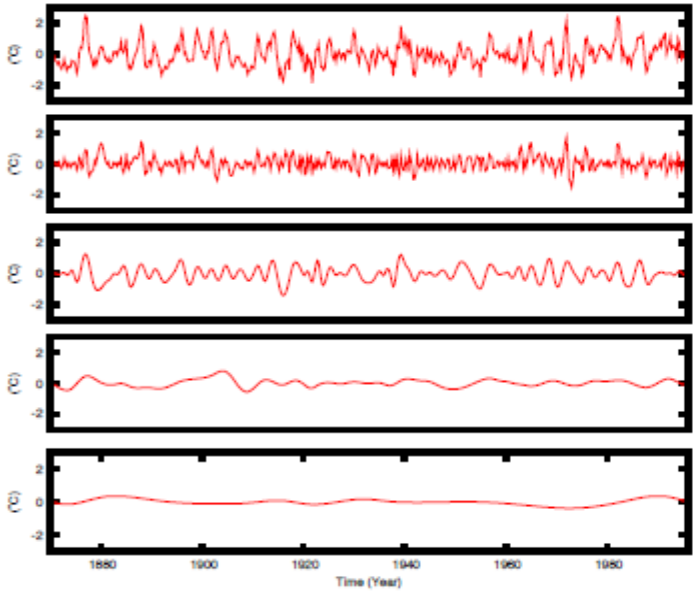
Original



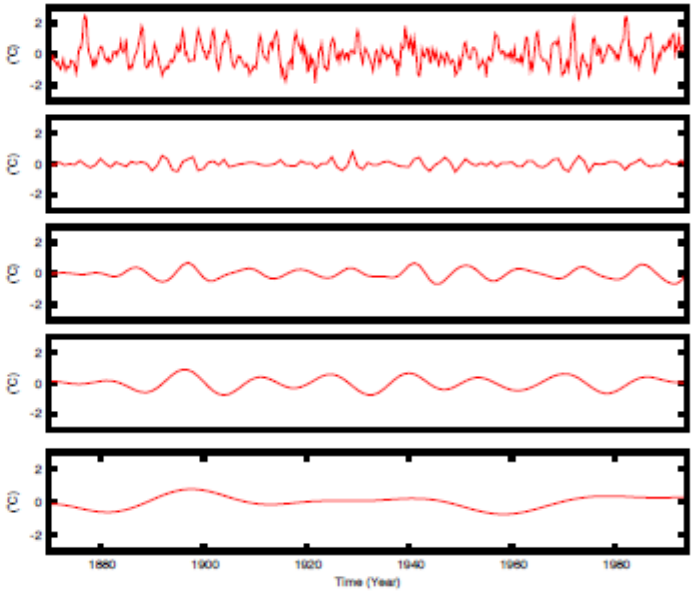
Enhanced

Central Pacific sea surface temperature

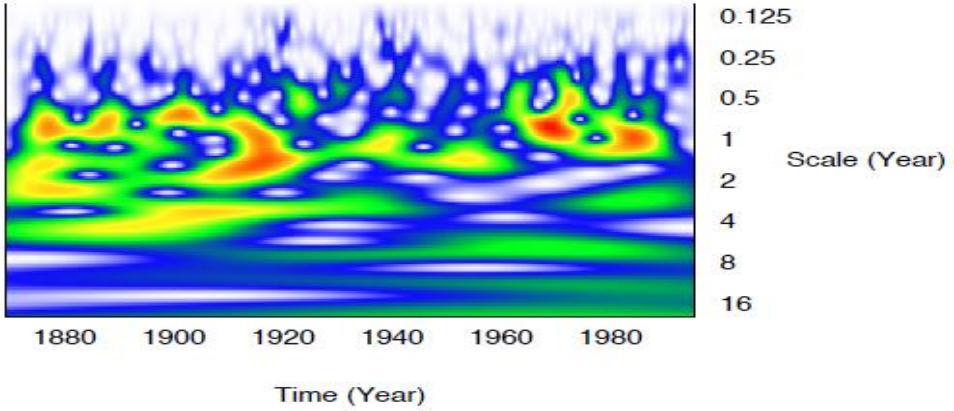




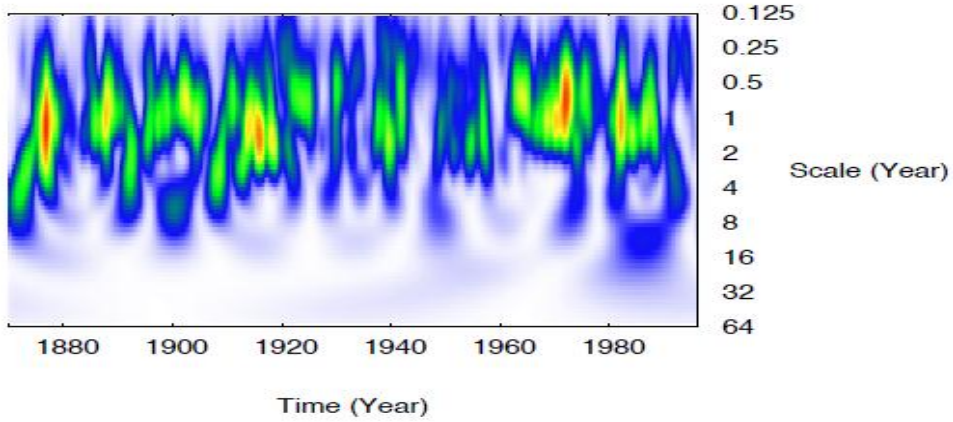
(a) The original SST signal (top panel) and the first four significant intrinsic mode functions generated by the EMD method.



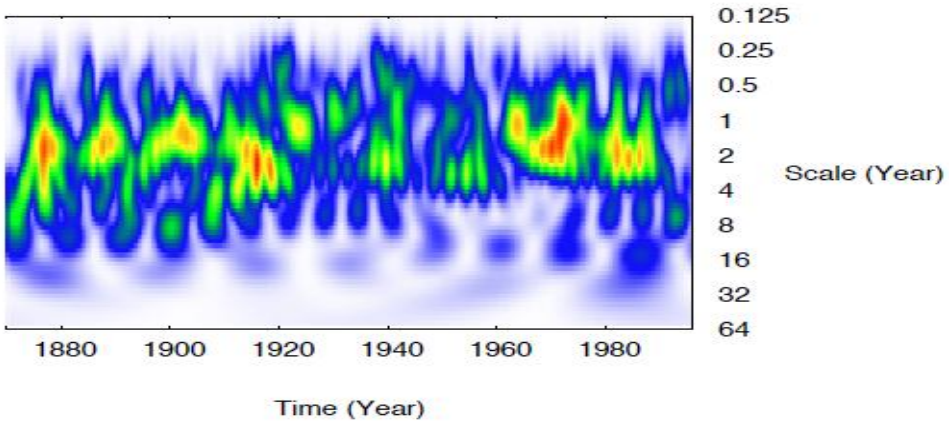
(b) The four modes generated by the PDE transform for the same SST signal on the top panel are similar to those from the EMD decomposition.



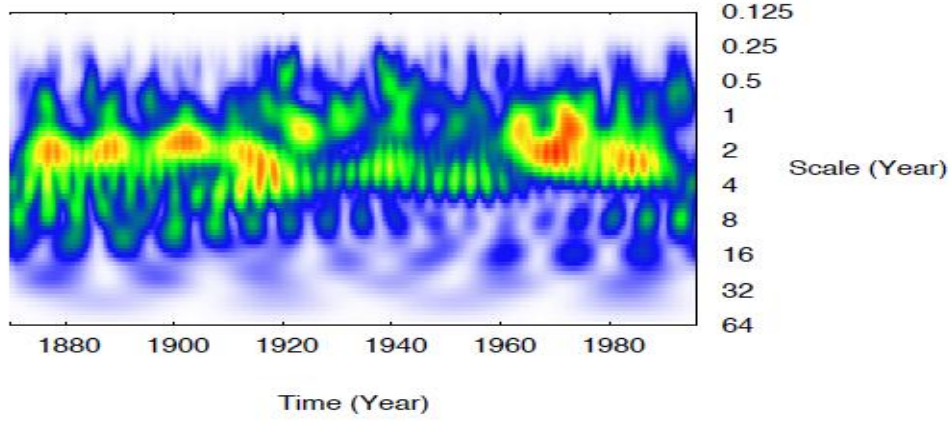
(a) Continuous wavelet analysis of the SST Nino3 data.



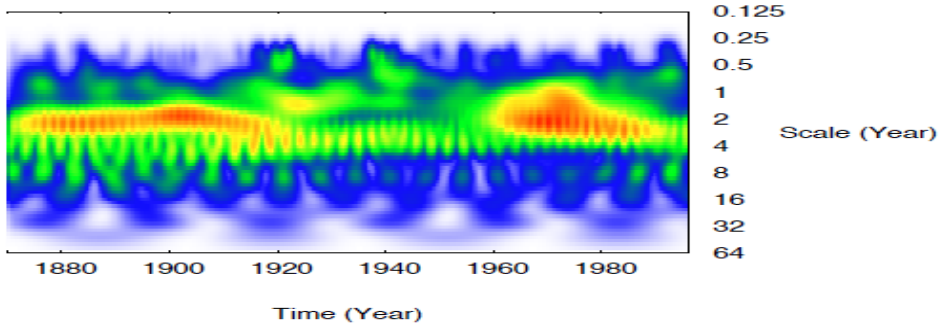
(b) PDE transform using 2nd order PDE.



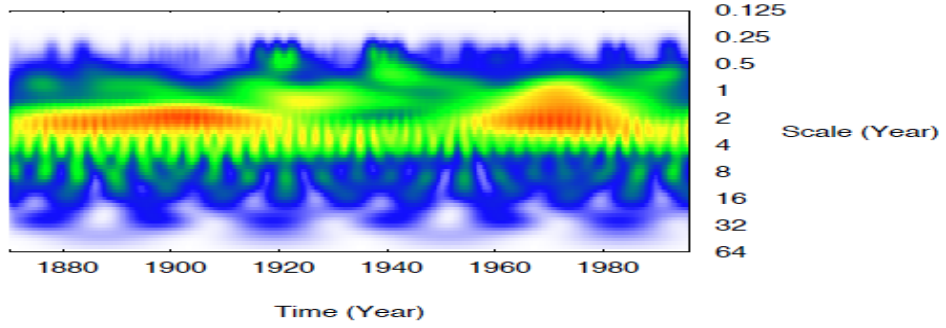
(c) PDE transform using 4th order PDE.



(d) PDE transform using 8th order PDE.

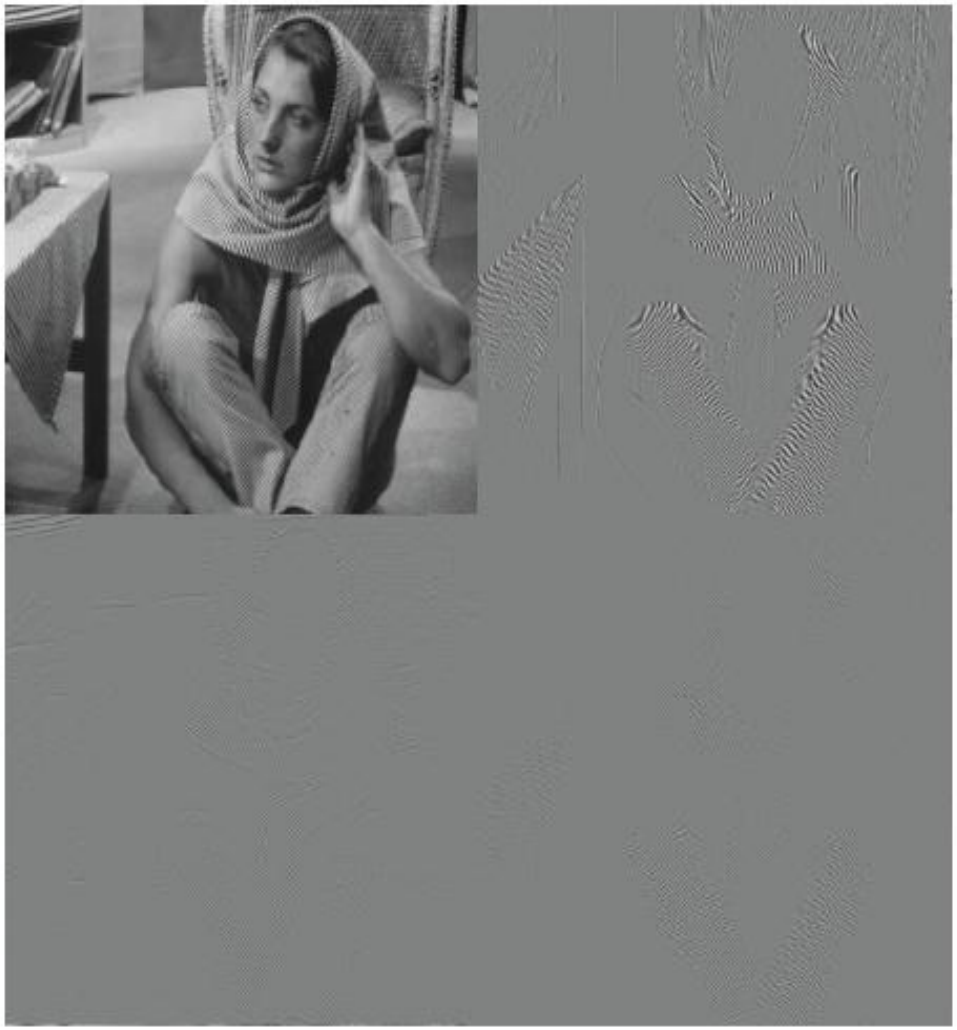


(e) PDE transform using 20th order PDE.



(f) PDE transform using 40th order PDE.

Wavelet transform vs PDE transform



(a) Discrete wavelet decomposition of the Lena image.

(b) PDE transform (using 2nd order PDE) of Barbara image.

PDE transform based local statistical analysis

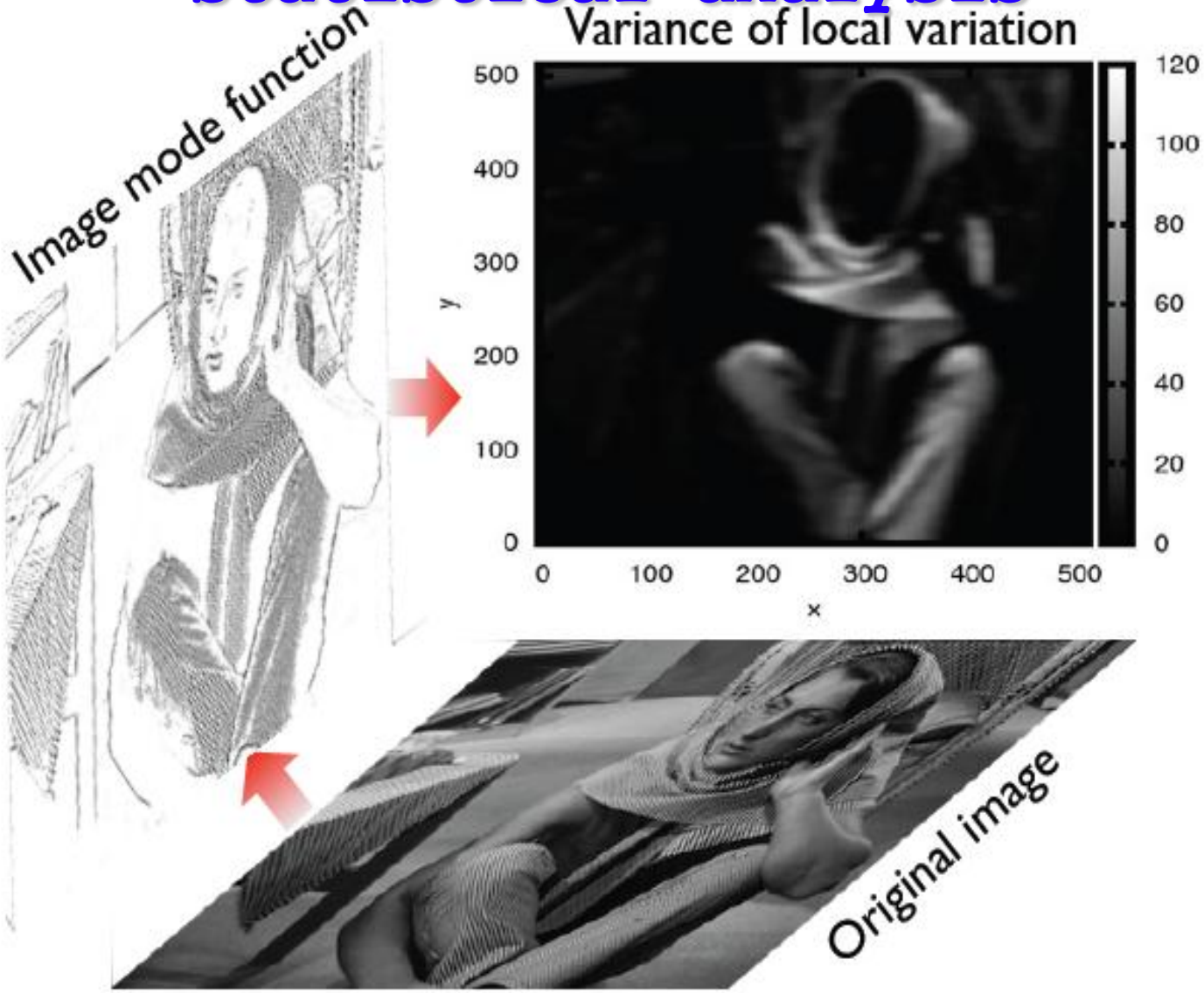
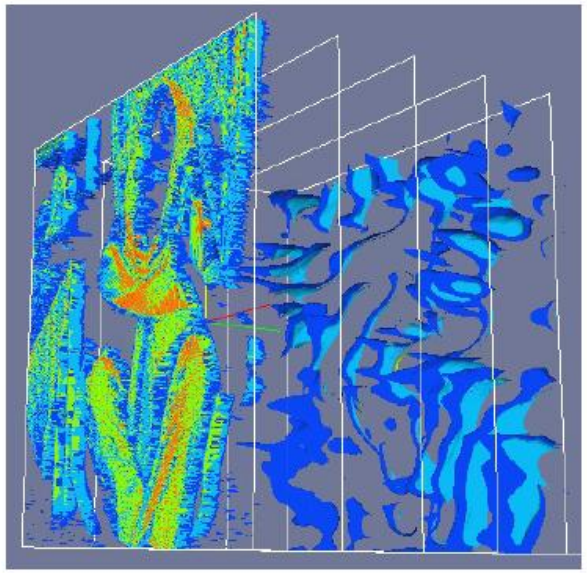


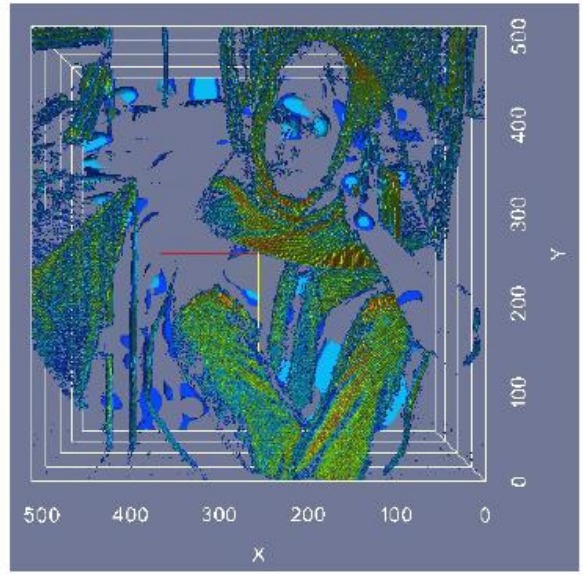
Figure 4: Adaptive PDE transform for selective texture extraction in the Barbara image. The variance of the local variation is shown in the top chart.

PDE transform based correlations

(a) Original Barbara image.

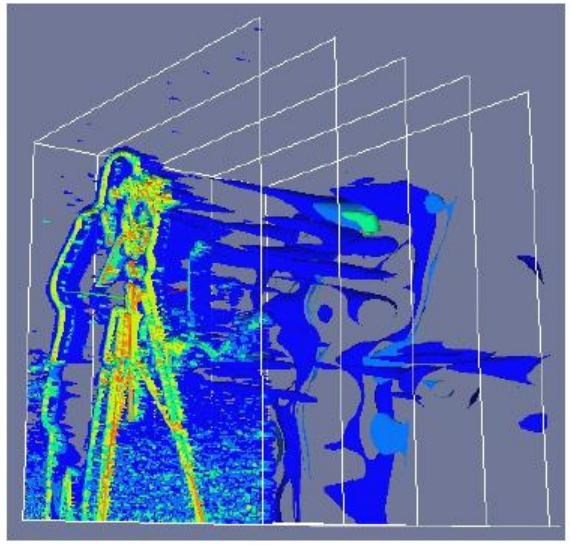


(b) 4D density of the Barbara image.

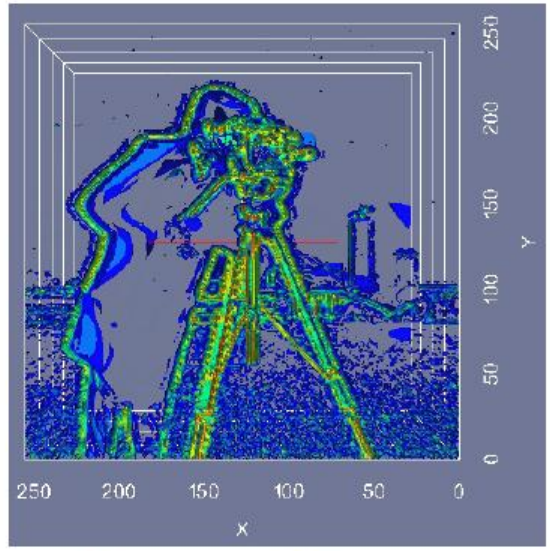


(c) Front view of the Figure 8(b).

(d) Original camera man image.



(e) 4D density of the camera man image.

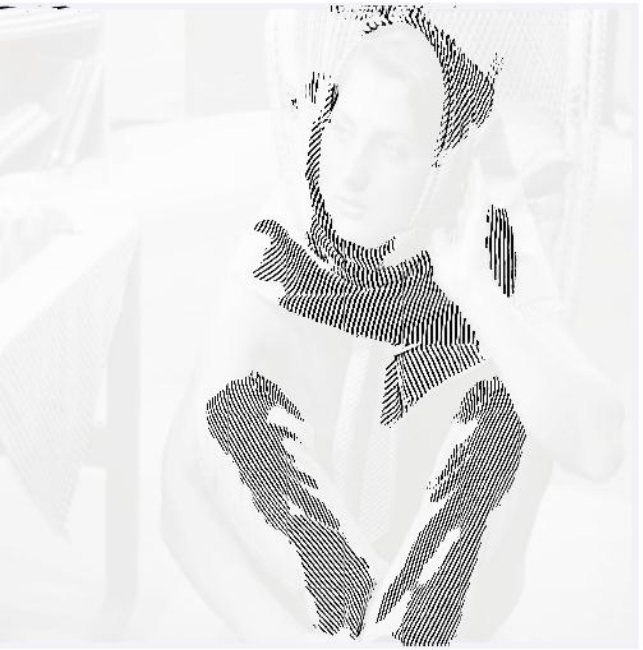


(f) Front view of the Figure 8(e).

(a) Original image.

(b) Image mode function.

(e) Texture 3



(c) Texture 1

(d) Texture 2

(f) Texture 4

PDE transform for texture analysis

PDE transform for texture analysis

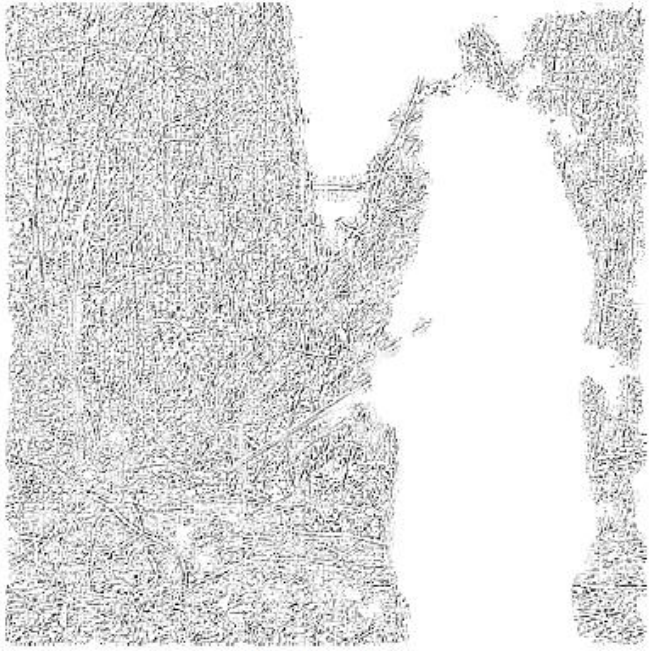
(a) Original image.

(b) Mode function.

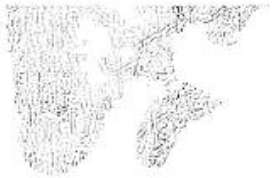
(c) Extracted texture.

Figure 3: Extraction and separation of texts, background watermark, and textures of image 3(a). Shown in the3(b) and 3(c) are the image mode function and extracted texture using the proposed adaptive PDE transform.

(a) Original image.



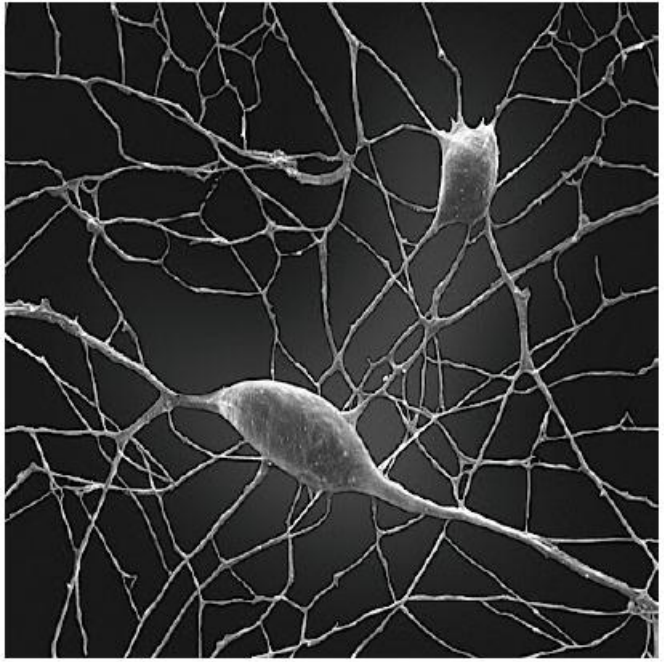
(b) Texture 1



(c) Texture 2

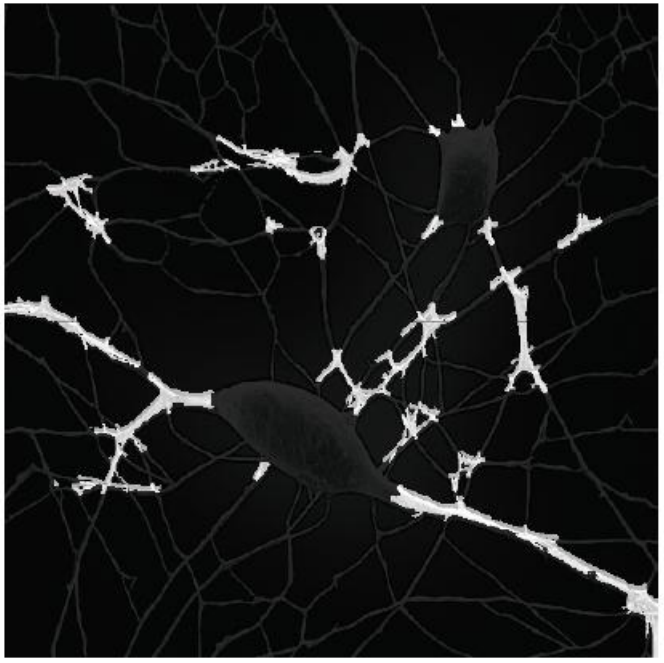
(d) Texture 3

PDE transform for sniper identification

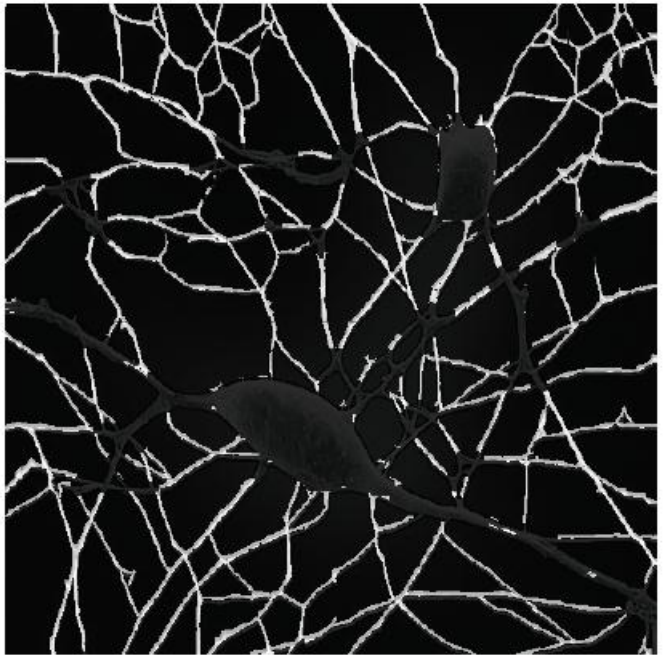


(a) Original neuron image.

(b) Class 1 of the selective neuron skeleton.



(c) Class 2 of the selective neuron skeleton.



(d) Class 3 of the selective neuron skeleton.

PDE transform for neuron classification

PDE transform for protein surface construction

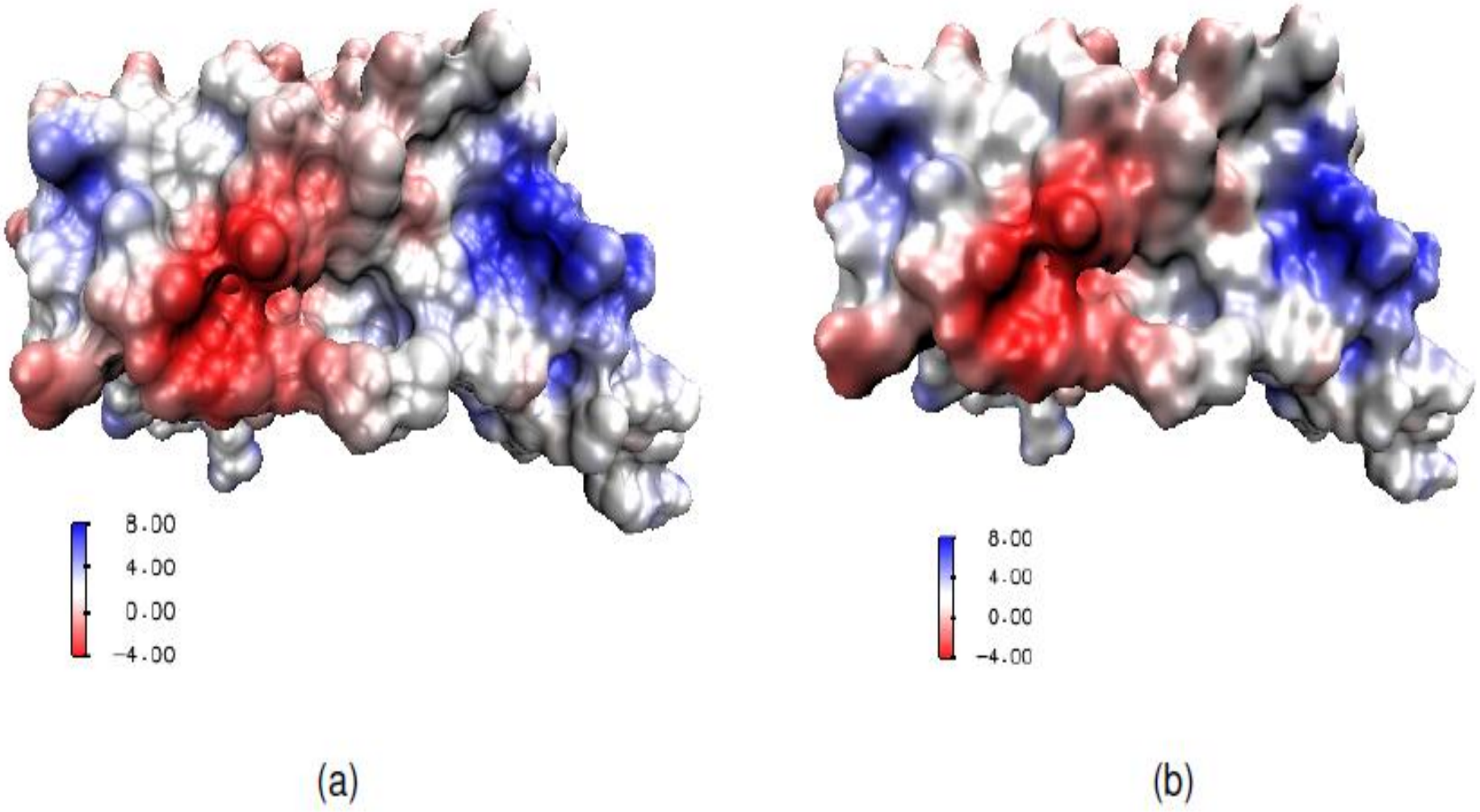
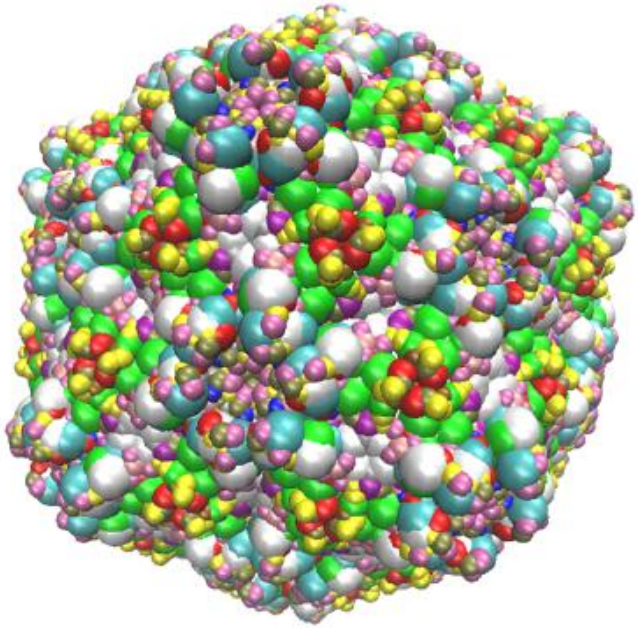
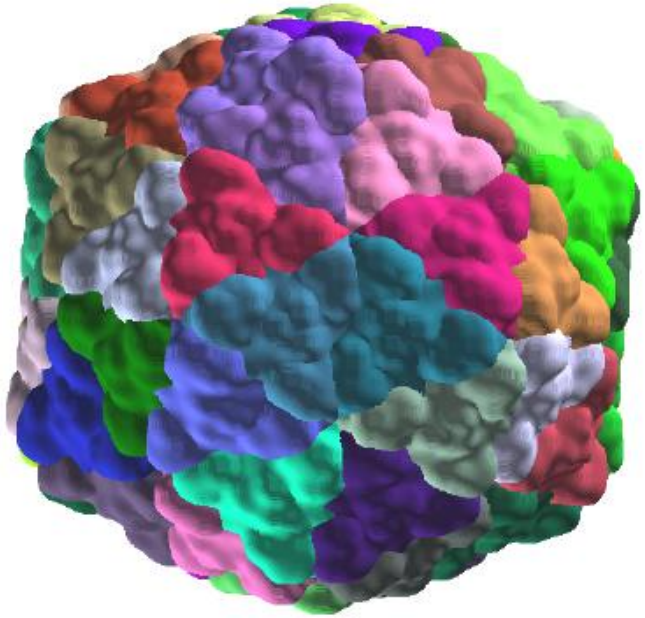


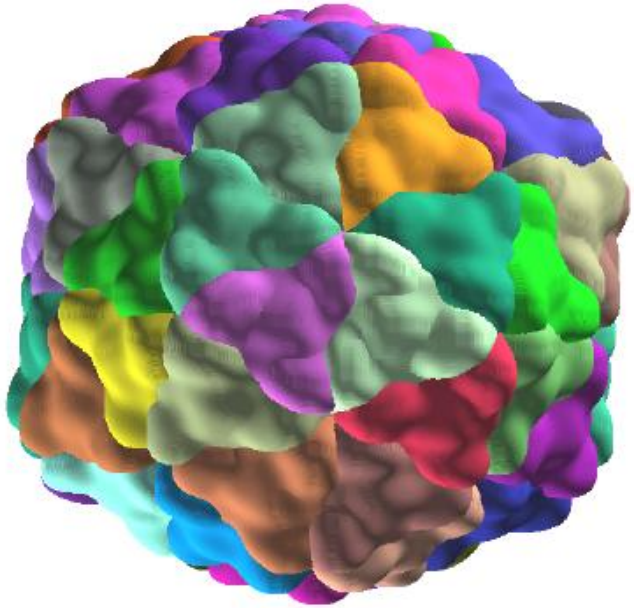
Figure 13: Electrostatic surface potentials for protein 1a7m mapped on two surfaces. (a) Surface generated by MSMS; (b) Surface generated by the 12th order PDE transform.



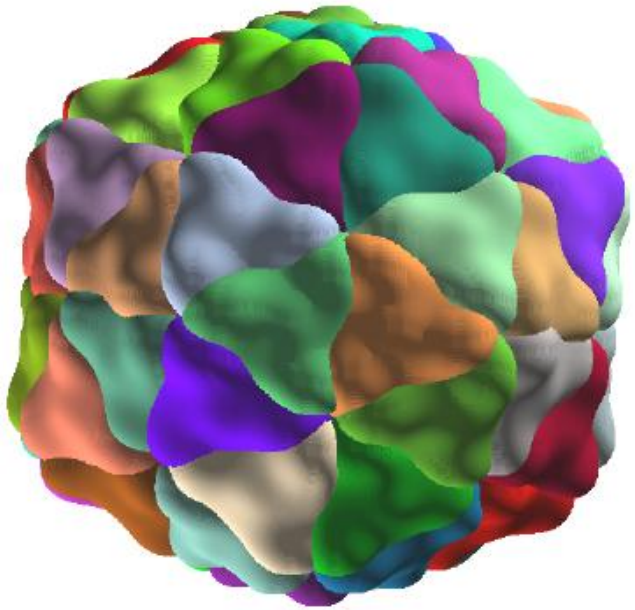
(a)



(b)

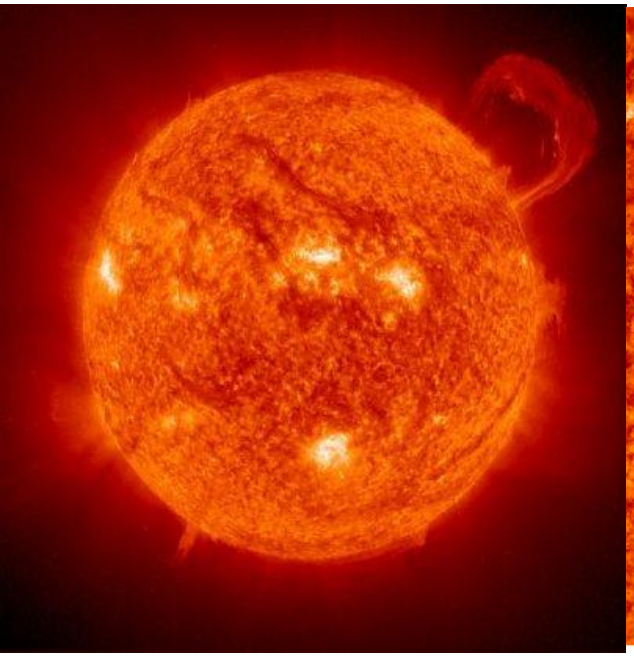


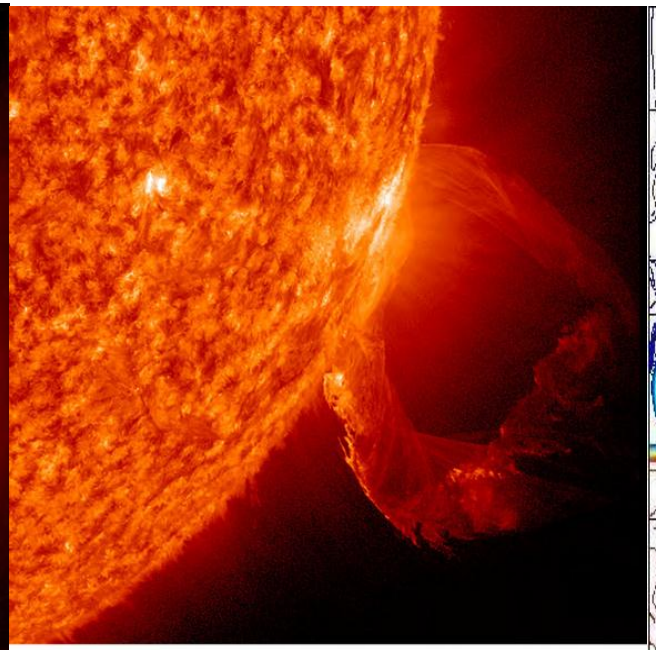
(c)

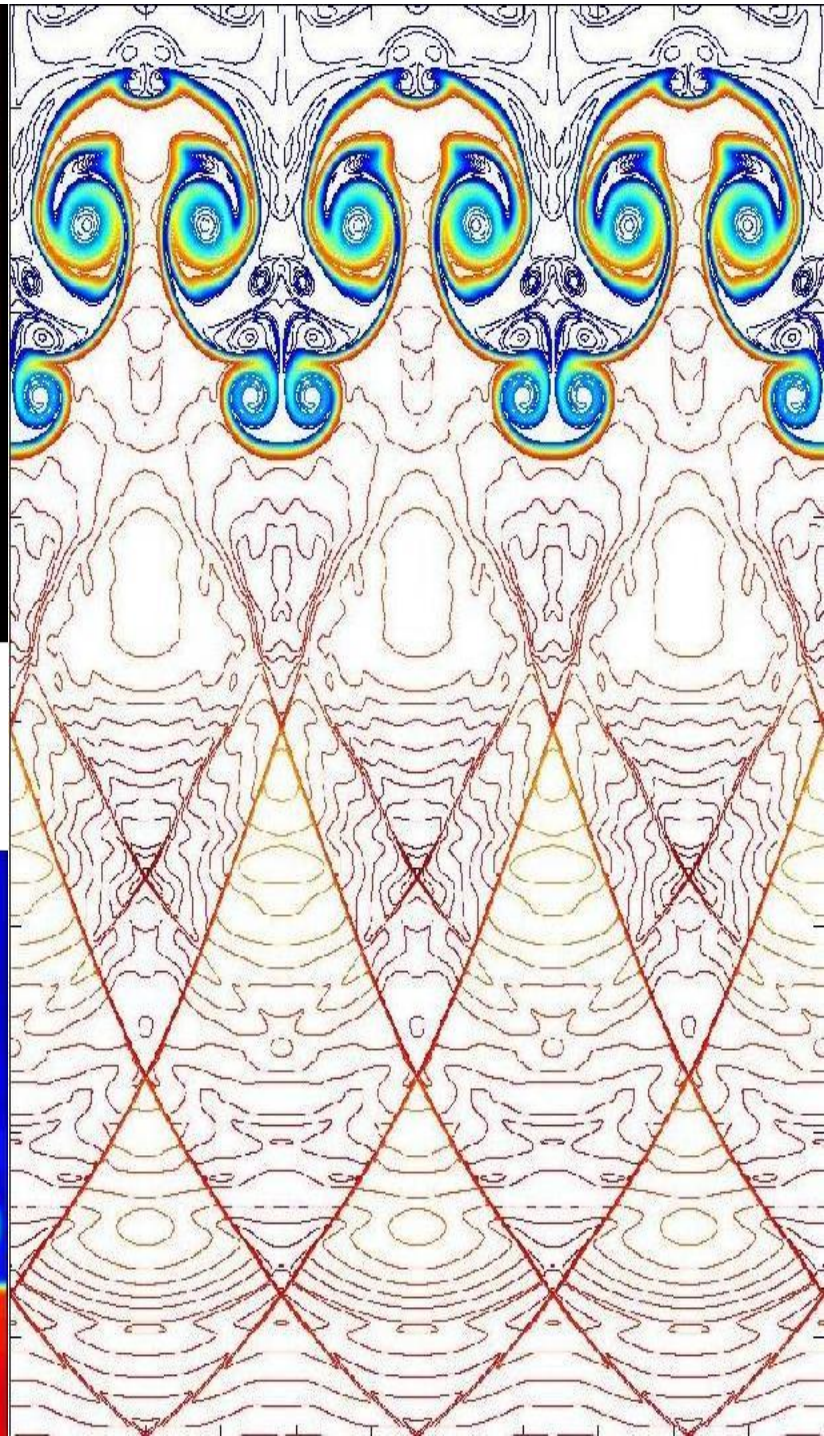


(d)

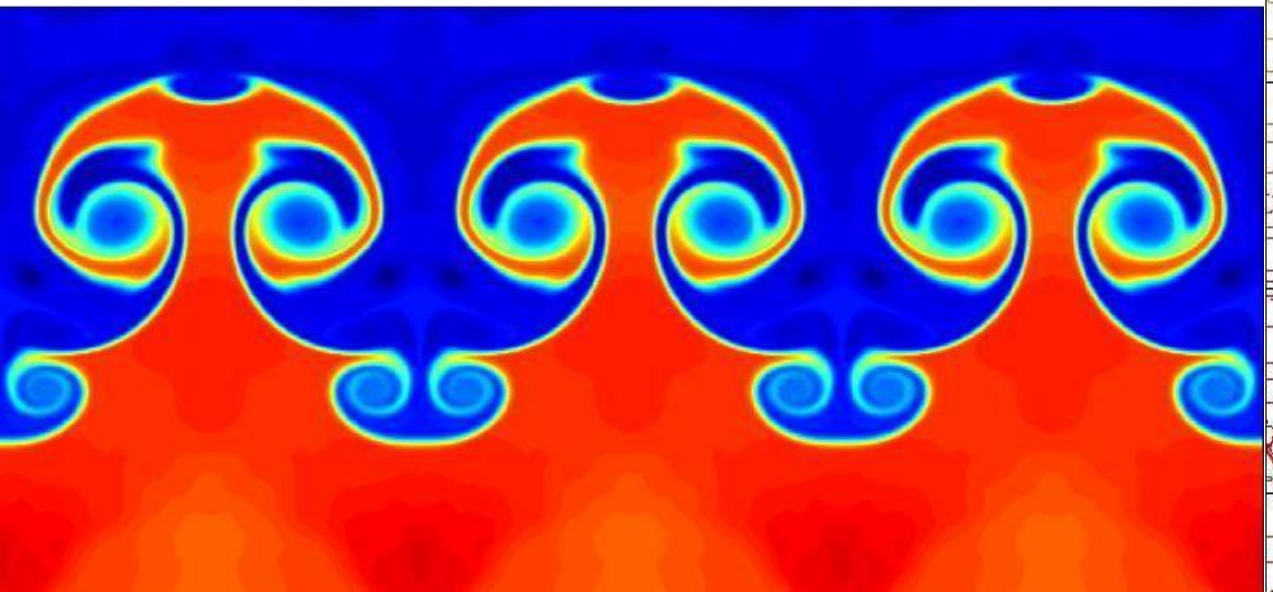
PDE transform for multiscale analysis







Richtmyer-Meshkov instability



Comparison of Hilbert-Huang, wavelet, Fourier and PDE transforms

- Only yield the relevant functional modes
- Each mode contains desired frequency range
- Mode is extracted using accurate high order PDEs based band-pass filters

- Each sub-band width is totally controllable
- Each mode function is determined by PDE order and evolution time
- Adjustable dual temporal-frequency localization

- Each mode contains selected frequency range
- Physical domain representation
- Applicable for non-stationary signal, and no Gibbs oscillations

Partial Differential Equation Transform

Hilbert-Huang transform

- Each mode is obtained by spline based lowpass filter
- Instantaneous frequency is obtained for characterizing non-stationary data

Wavelet transform

- Dual time-scale analysi
- Robust choice of the mother wavelet
- Dilation and translation are used to capture the local characteristics

Fourier transform

- Perfect localization in frequency space
- Gibbs oscillations
- Impressive improvements and applications are still occurring